Polynomial Functions and Their Graphs

Definition of a Polynomial Function

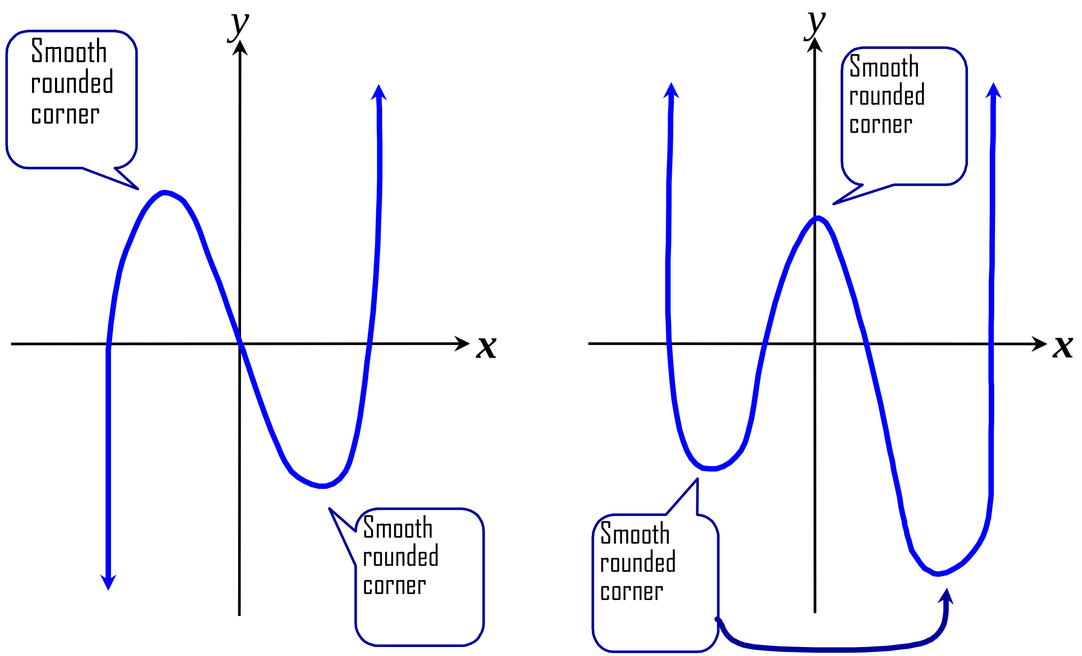
Let n be a nonnegative integer and let a_n, a_{n-1},…, a_2, a_1, a_0, be real numbers with $a_n \neq 0$. The function defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of *x* of degree *n*.** The number a_n, the coefficient of the variable to the highest power, is called the **leading coefficient**.

Smooth, Continuous Graphs

Two important features of the graphs of polynomial functions are that they are smooth and *continuous*. By **smooth,** we mean that the graph contains only rounded curves with no sharp corners. By **continuous,** we mean that the graph has no breaks and can be drawn without lifting your pencil from the rectangular coordinate system. These ideas are illustrated in the figure.



The Leading Coefficient Test

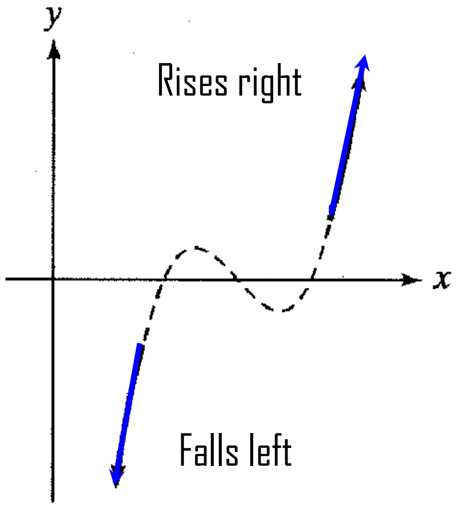
As x increases or decreases without bound, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 \quad (a_n \neq 0)$$

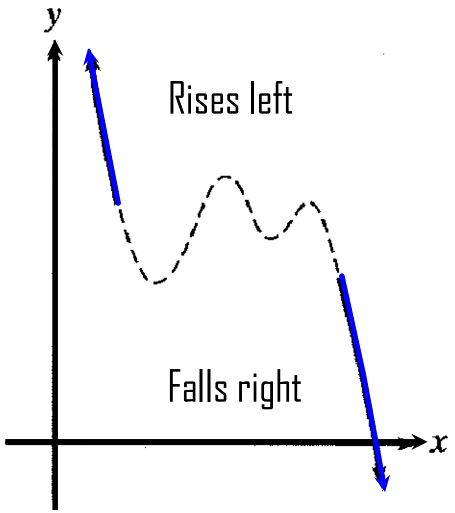
eventually rises or falls. In particular,

1. For n odd: $a_n > 0$ $a_n < 0$

If the leading coefficient is positive, the graph falls to the left and rises to the right.



If the leading coefficient is negative, the graph rises to the left and falls to the right.



The Leading Coefficient Test

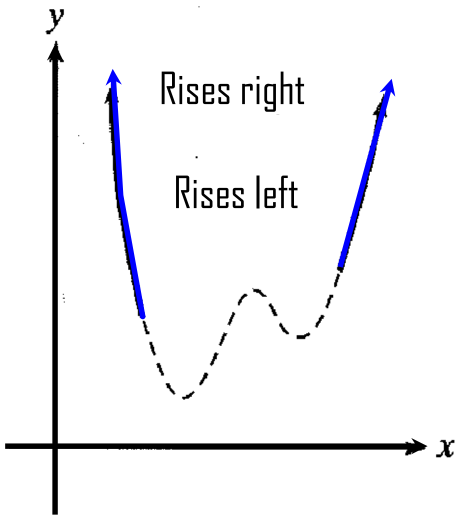
As x increases or decreases without bound, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 \quad (a_n \neq 0)$$

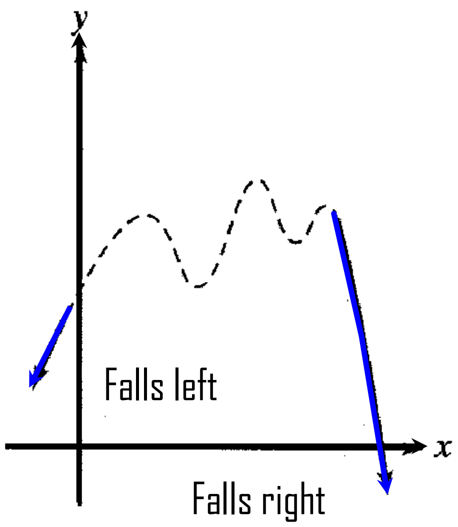
eventually rises or falls. In particular,

1. For n even: $a_n > 0$ $a_n < 0$

If the leading coefficient is positive, the graph rises to the left and to the right.



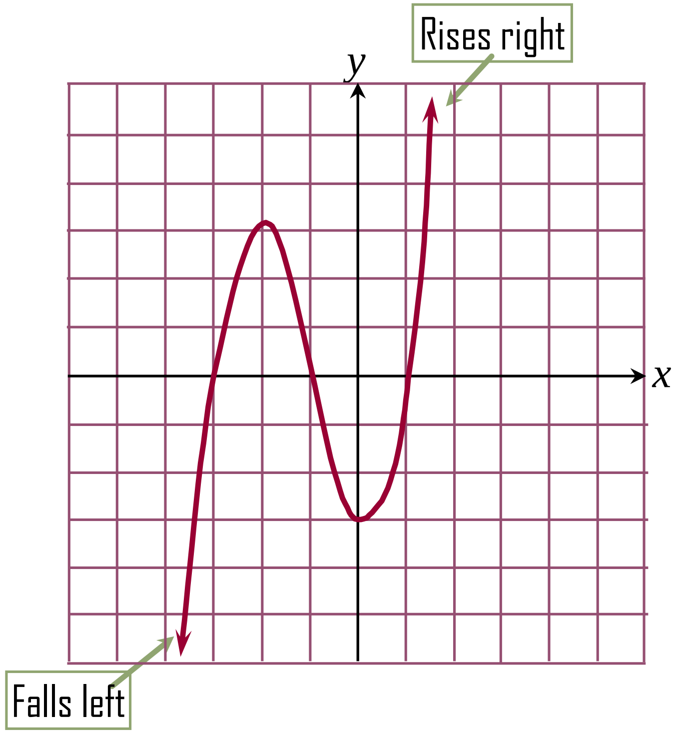
If the leading coefficient is negative, the graph falls to the left and to the right.



Text Example

Use the Leading Coefficient Test to determine the end behavior of the graph of Graph the quadratic function $f(x) = x^3 + 3x^2 - x - 3$.

Solution Because the degree is odd ($n = 3$) and the leading coefficient, 1, is positive, the graph falls to the left and rises to the right, as shown in the figure.



Text Example

Find all zeros of $f(x) = -x^4 + 4x^3 - 4x^2$.

Solution We find the zeros of f by setting $f(x)$ equal to 0.

$-x^4 + 4x^3 - 4x^2 = 0$ **We now have a polynomial equation.**

$x^4 - 4x^3 + 4x^2 = 0$ **Multiply both sides by –1. (optional step)**

$x^2(x^2 - 4x + 4) = 0$ **Factor out x^2.**

$x^2(x - 2)^2 = 0$ **Factor completely.**

$x^2 = 0$ or $(x - 2)^2 = 0$ **Set each factor equal to zero.**

$x = 0$ $x = 2$ **Solve for x.**

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Multiplicity and x-Intercepts

If *r* is a zero of even multiplicity, then the graph **touches** the *x*-axis and turns around at *r*. If *r* is a zero of odd multiplicity, then the graph **crosses** the *x*-axis at *r*. Regardless of whether a zero is even or odd, graphs tend to flatten out at zeros with multiplicity greater than one.

Example

- Find the x-intercepts and multiplicity of $f(x) = 2(x+2)^2(x-3)$

Solution:

- $x=-2$ is a zero of multiplicity 2 or even
- $x=3$ is a zero of multiplicity 1 or odd

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Graphing a Polynomial Function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \quad (a_n \neq 0)$$

1. Use the Leading Coefficient Test to determine the graph's end behavior.
2. Find x-intercepts by setting $f(x) = 0$ and solving the resulting polynomial equation. If there is an x-intercept at r as a result of $(x - r)^k$ in the complete factorization of $f(x)$, then:
   a. If k is even, the graph touches the x-axis at r and turns around.
   b. If k is odd, the graph crosses the x-axis at r.
   c. If $k > 1$, the graph flattens out at $(r, 0)$.

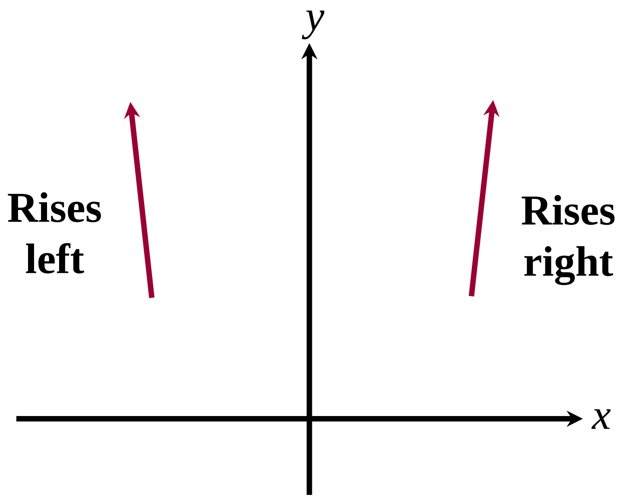
3. Find the y-intercept by setting x equal to 0 and computing $f(0)$.

Example

Graph: $f(x) = x^4 - 2x^2 + 1$.

Solution

Step 1 **Determine end behavior.** Because the degree is even ($n = 4$) and the leading coefficient, 1, is positive, the graph rises to the left and the right:



Example cont.

Graph: $f(x) = x^4 - 2x^2 + 1$.

Solution

Step 2 **Find the x-intercepts (zeros of the function) by setting $f(x) = 0$.**

$$x^4 - 2x^2 + 1 = 0$$

$$(x^2 - 1)(x^2 - 1) = 0$$ **Factor.**

$$(x + 1)(x - 1)(x + 1)(x - 1) = 0$$ **Factor completely.**

$$(x + 1)^2(x - 1)^2 = 0$$ **Express the factoring in more compact notation.**

$(x + 1)^2 = 0$ or $(x - 1)^2 = 0$ **Set each factor equal to zero.**

$x = -1$ $\quad$ $x = 1$ **Solve for x.**

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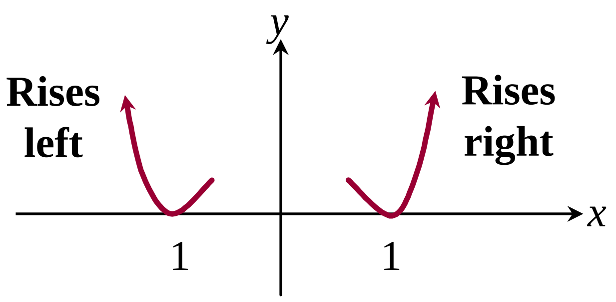
Example cont.

Graph: $f(x) = x^4 - 2x^2 + 1$.

Solution

Step 2 We see that -1 and 1 are both repeated zeros with multiplicity 2. Because of the even multiplicity, the graph touches the *x*-axis at –1 and 1 and turns around. Furthermore, the graph tends to flatten out at these zeros with multiplicity greater than one:



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Example cont.

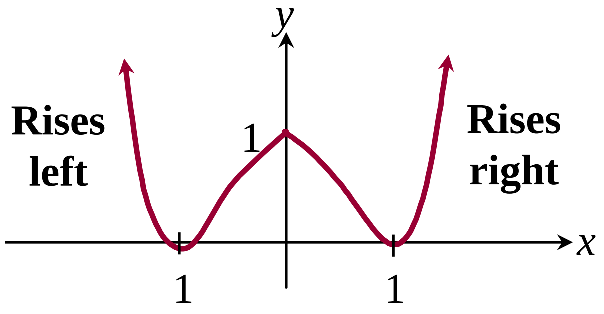
Graph: $f(x) = x^4 - 2x^2 + 1$.

Solution

Step 3 **Find the *y*-intercept.** Replace x with 0 in $f(x) = -x + 4x - 1$.

$$f(0) = 0^4 - 2 \bullet 0^2 + 1 = 1$$

There is a *y*-intercept at 1, so the graph passes through (0, 1).



Example cont.

Graph: $f(x) = x^4 - 2x^2 + 1$.

Solution