



James Madison  
HIGH SCHOOL

# UNIT CIRCLE

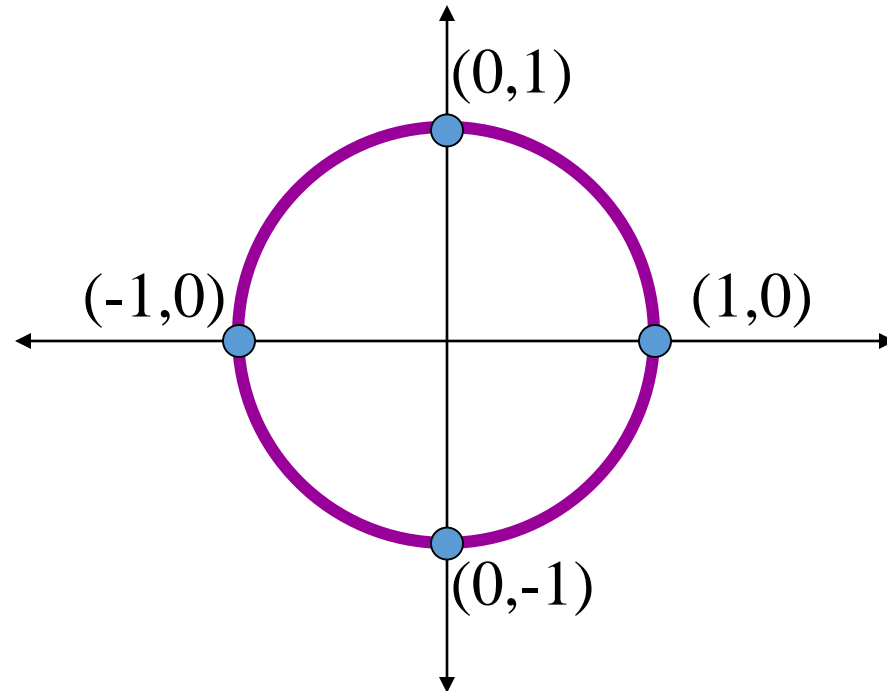
# THE YEAR



A circle with center at  $(0, 0)$  and radius 1 is called a unit circle.

The equation of this circle would be

$$x^2 + y^2 = 1$$

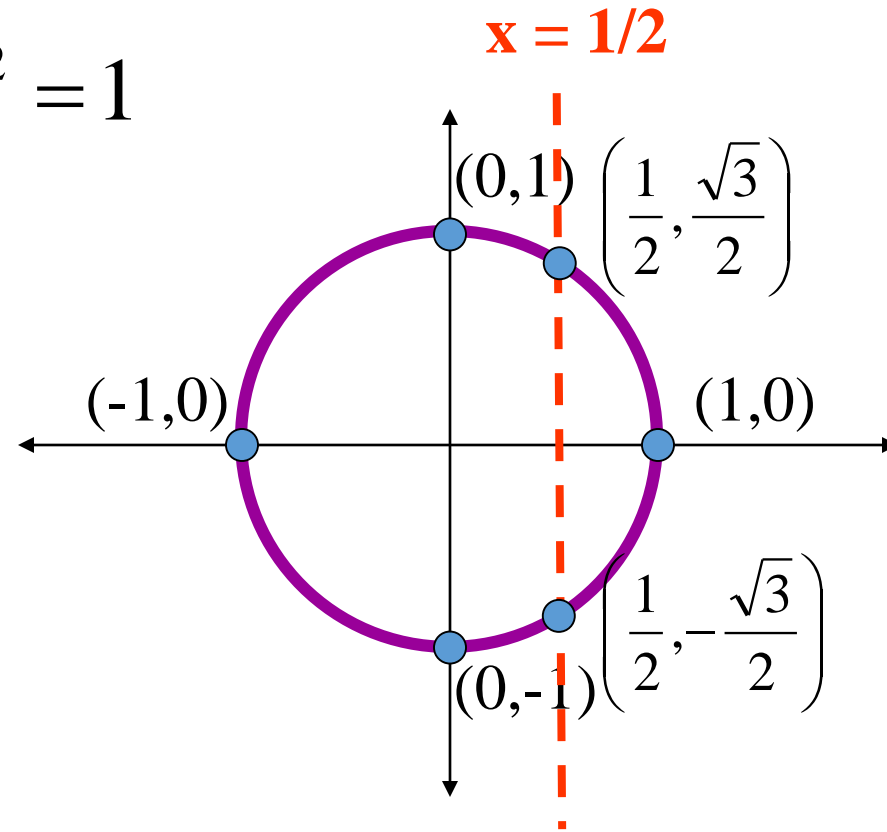


So points on this circle must satisfy this equation.



Let's pick a point on the circle. We'll choose a point where the  $x$  is  $1/2$ . If the  $x$  is  $1/2$ , what is the  $y$  value?

$$x^2 + y^2 = 1$$
$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$
$$y^2 = \frac{3}{4}$$
$$y = \pm \frac{\sqrt{3}}{2}$$

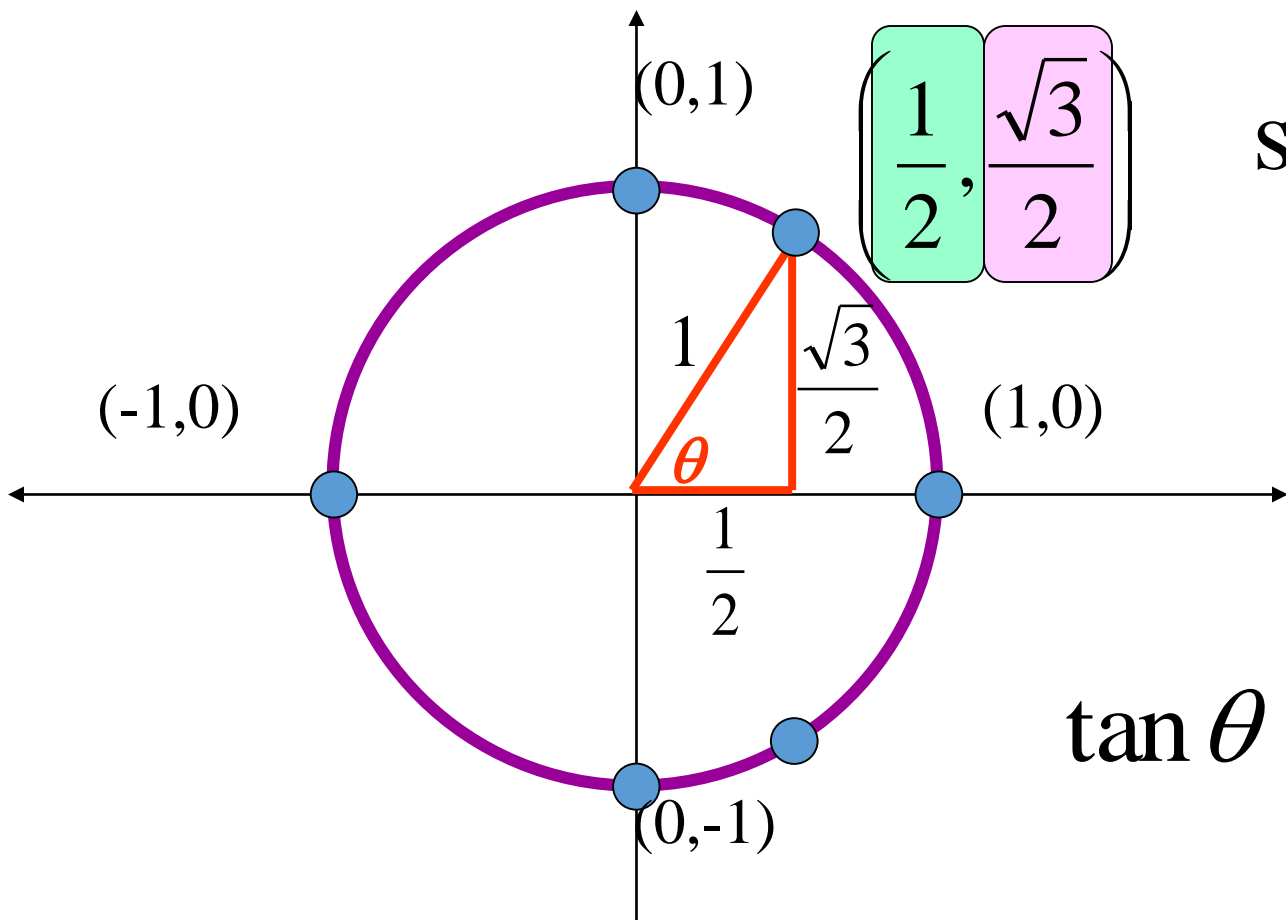


You can see there are two  $y$  values. They can be found by putting  $1/2$  into the equation for  $x$  and solving for  $y$ .

We'll look at a larger version of this and make a right triangle.



We know all of the sides of this triangle. The bottom leg is just the x value of the point, the other leg is just the y value and the hypotenuse is always 1 because it is a radius of the circle.



$$\sin \theta = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

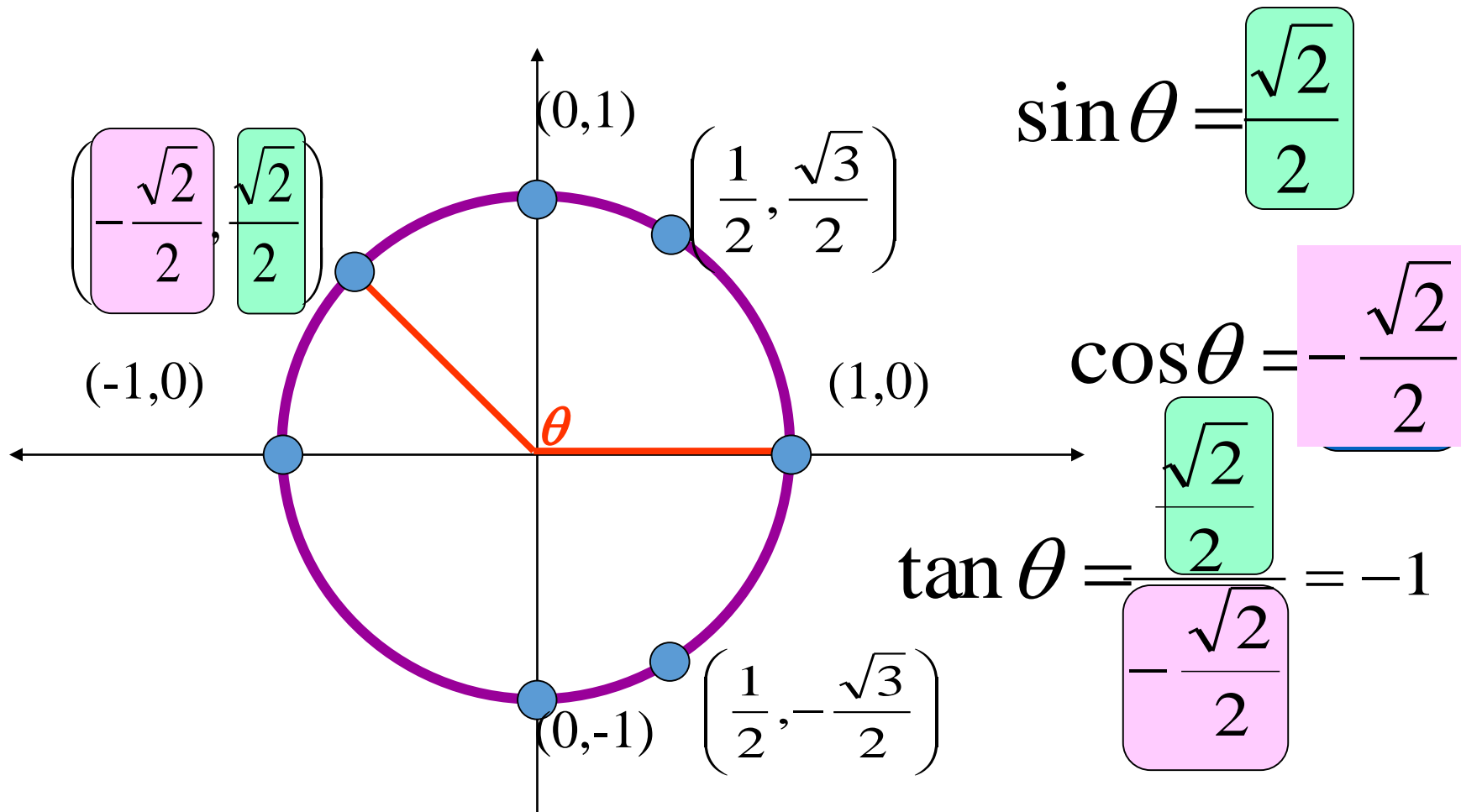
$$\cos \theta = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\tan \theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Notice the sine is just the y value of the unit circle point and the cosine is just the x value.



So if I want a trig function for  $\theta$  whose terminal side contains a point on the unit circle, the y value is the sine, the x value is the cosine and  $y/x$  is the tangent.

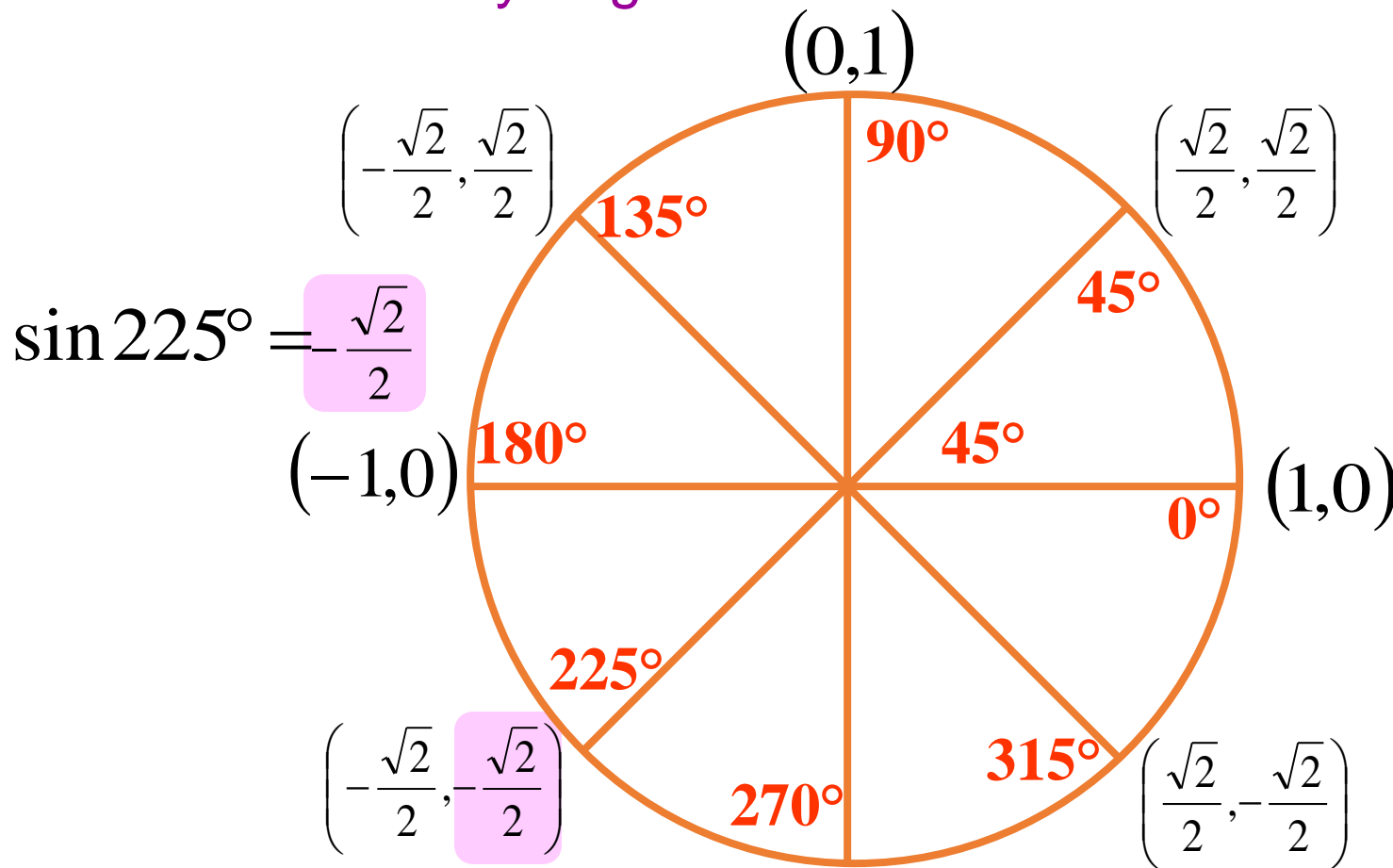


We divide the unit circle into various pieces and learn the point values so we can then from memory find trig functions.



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Here is the unit circle divided into 8 pieces. Can you figure out how many degrees are in each division?

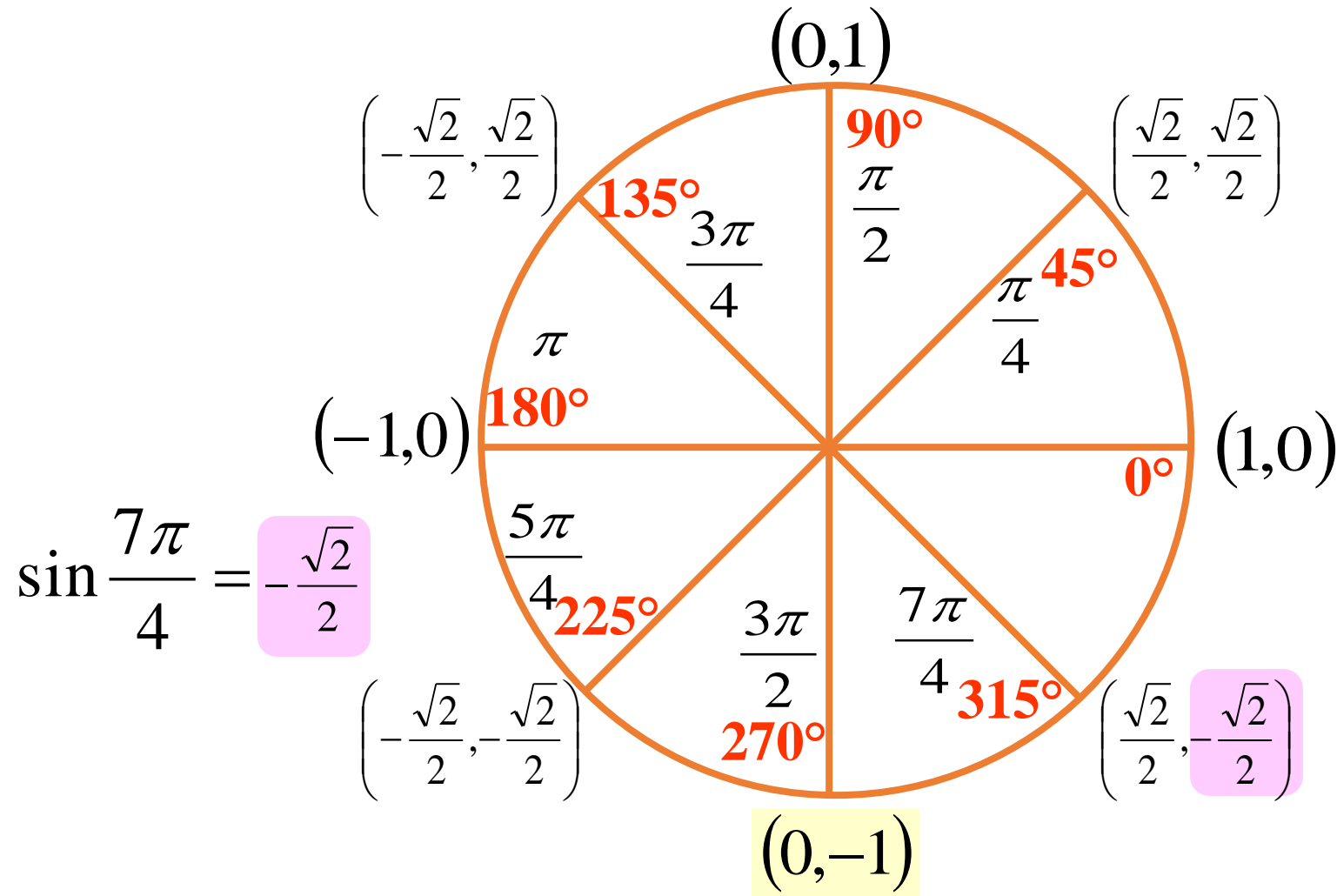


These are easy to memorize since they all have the same value with different signs depending on the quadrant.

We can label this all the way around with how many degrees an angle would be and the point on the unit circle that corresponds with the terminal side of the angle. We could then find any of the trig functions.



Can you figure out what these angles would be in radians?



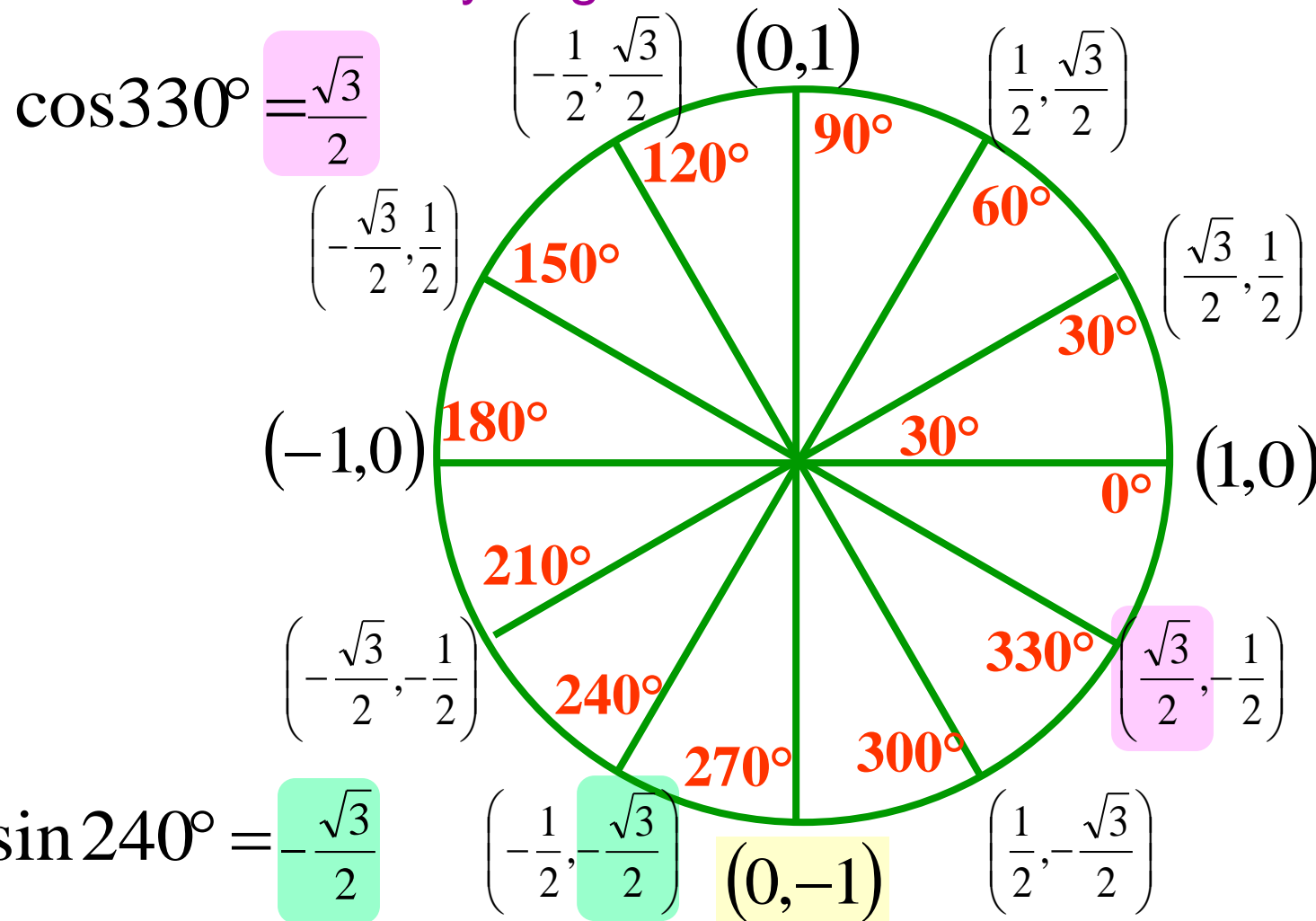
The circle is  $2\pi$  all the way around so half way is  $\pi$ . The upper half is divided into 4 pieces so each piece is  $\pi/4$ .



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Here is the unit circle divided into 12 pieces. Can you figure out how many degrees are in each division?

You'll need to memorize these too but you can see the pattern.

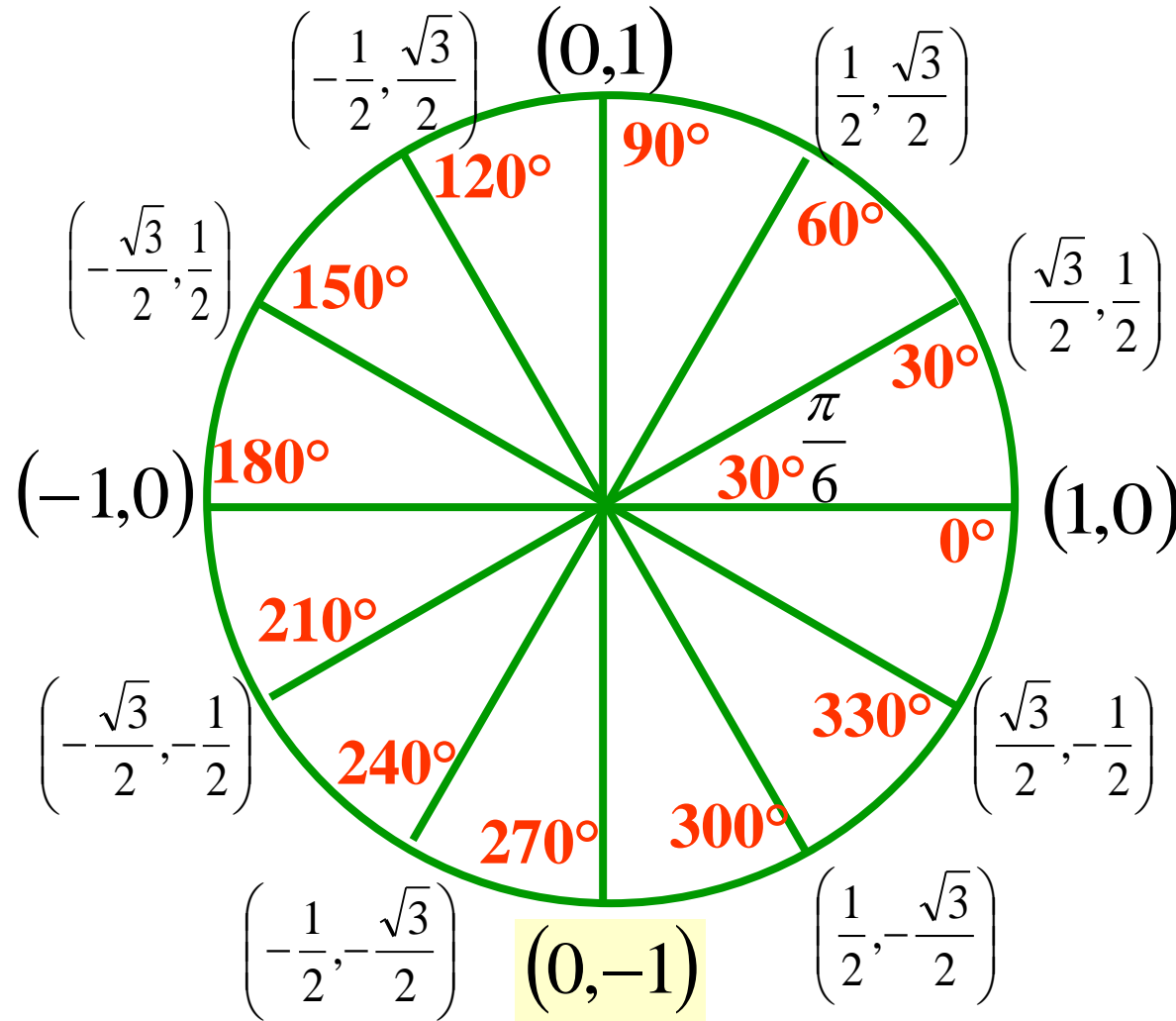


We can again label the points on the circle and the sine is the y value, the cosine is the x value and the tangent is y over x.





Can you figure out what the angles would be in radians?



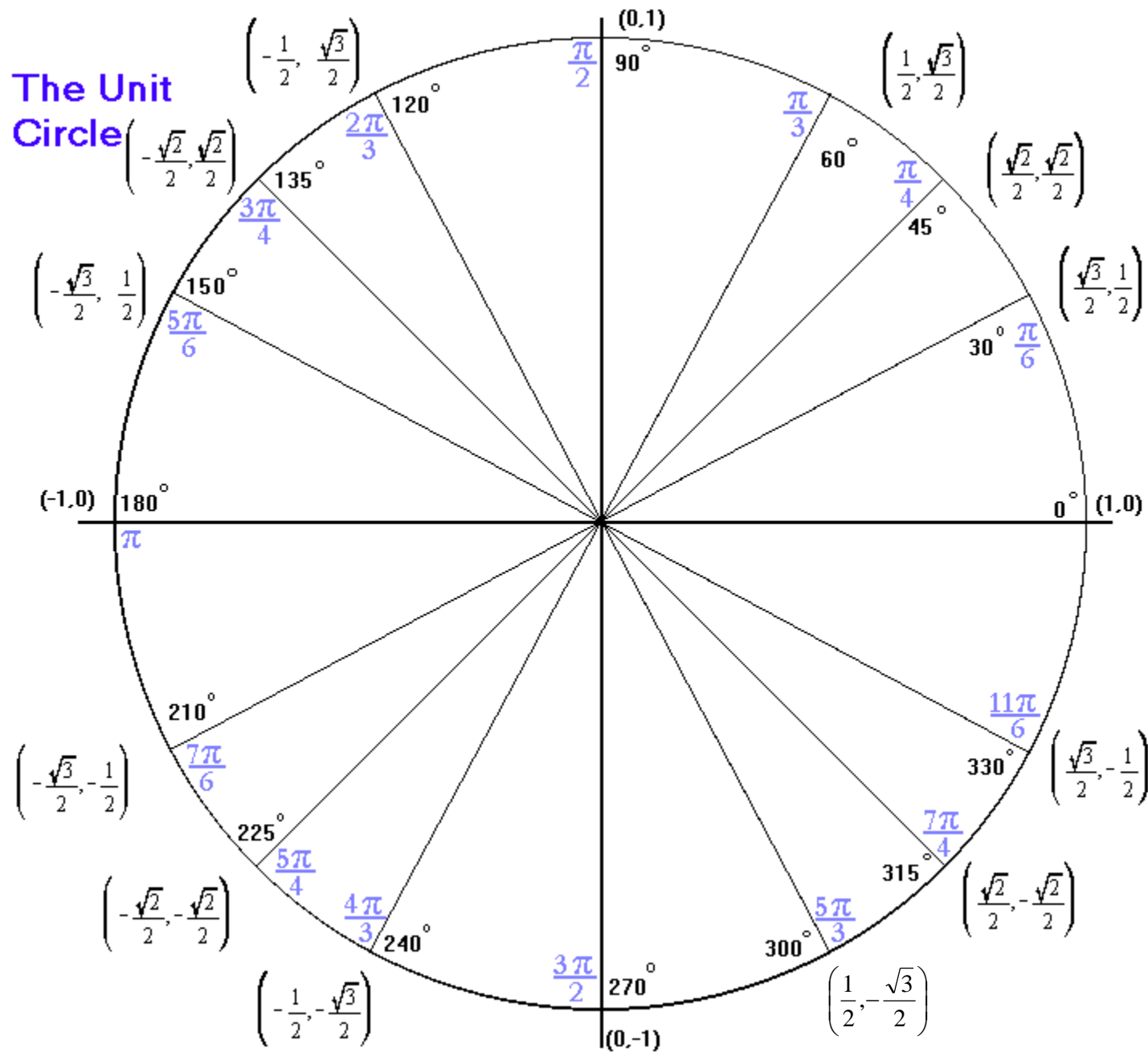
We'll see them all put together on the unit circle on the next screen.

It is still  $\pi$  halfway around the circle and the upper half is divided into 6 pieces so each piece is  $\pi/6$ .



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You should memorize this. This is a great reference because you can figure out the trig functions of all these angles quickly.





## Let's think about the function $f(\theta) = \sin \theta$

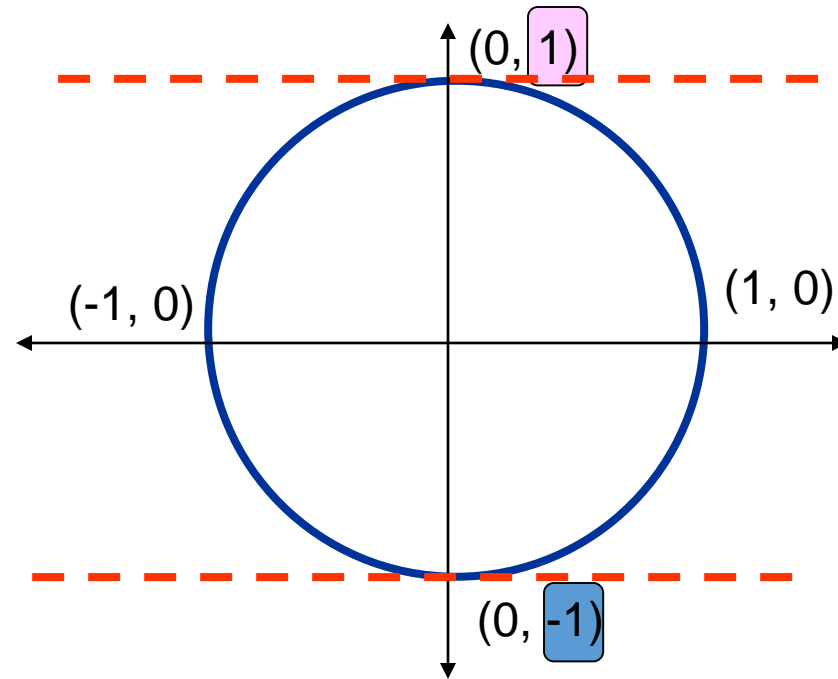
What is the domain? (remember domain means the “legal” things you can put in for  $\theta$ ).

You can put in anything you want so the domain is all real numbers.

What is the range? (remember range means what you get out of the function).

The range is:  $-1 \leq \sin \theta \leq 1$

Let's look at the unit circle to answer that. What is the **lowest** and **highest** value you'd ever get for sine? (sine is the y value so what is the lowest and highest y value?)





## Let's think about the function $f(\theta) = \cos \theta$

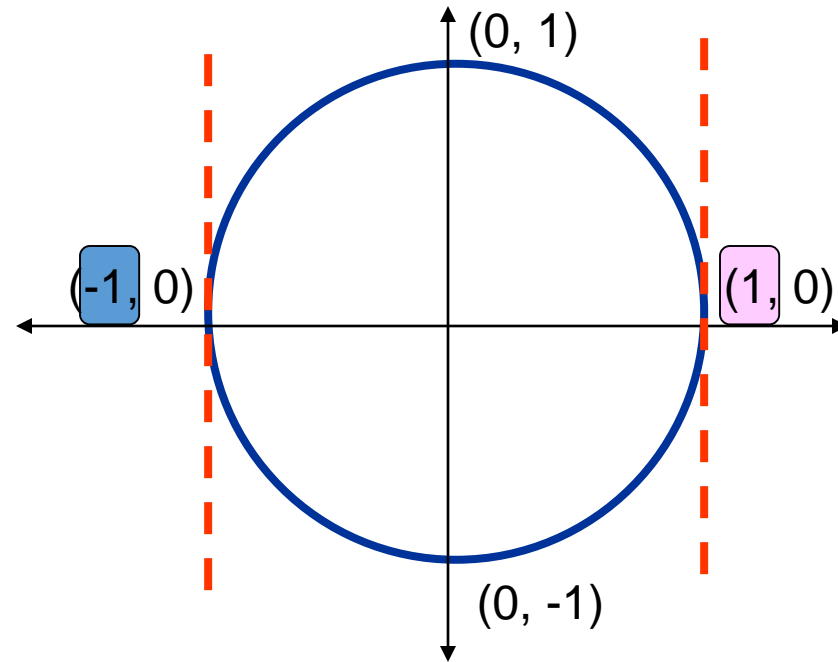
What is the domain? (remember domain means the “legal” things you can put in for  $\theta$ ).

You can put in anything you want so the domain is all real numbers.

What is the range? (remember range means what you get out of the function).

The range is:  $-1 \leq \cos \theta \leq 1$

Let's look at the unit circle to answer that. What is the **lowest** and **highest** value you'd ever get for cosine? (cosine is the  $x$  value so what is the lowest and highest  $x$  value?)





## Let's think about the function $f(\theta) = \tan \theta$

What is the domain? (remember domain means the “legal” things you can put in for  $\theta$ ).

Tangent is  $y/x$  so we will have an “illegal” if  $x$  is 0.  $x$  is 0 at  $90^\circ$  (or  $\pi/2$ ) or any odd multiple of  $90^\circ$

The domain then is all real numbers except odd multiples of  $90^\circ$  or  $\pi/2$ .

What is the range? (remember range means what you get out of the function).

If we take any  $y/x$ , we could end up getting any value so range is all real numbers.



## Let's think about the function $f(\theta) = \csc \theta$

What is the domain?

Since this is  $1/\sin \theta$ , we'll have trouble if  $\sin \theta = 0$ . That will happen at 0 and multiples of  $\pi$  (or  $180^\circ$ ). The domain then is all real numbers except multiples of  $\pi$ .

What is the range?

Since the range is:  $-1 \leq \sin \theta \leq 1$ , sine will be fractions less than one. If you take their reciprocal you will get things greater than 1. The range then is all real numbers greater than or equal to 1 or all real numbers less than or equal to -1.



## Let's think about the function $f(\theta) = \sec \theta$

What is the domain?

Since this is  $1/\cos \theta$ , we'll have trouble if  $\cos \theta = 0$ . That will happen at odd multiples of  $\pi/2$  (or  $90^\circ$ ). The domain then is all real numbers except odd multiples of  $\pi/2$ .

What is the range?

Since the range is:  $-1 \leq \cos \theta \leq 1$ , cosine will be fractions less than one. If you take their reciprocal you will get things greater than 1. The range then is all real numbers greater than or equal to 1 or all real numbers less than or equal to -1.



## Let's think about the function $f(\theta) = \cot \theta$

What is the domain?

Since this is  $\cos \theta / \sin \theta$ , we'll have trouble if  $\sin \theta = 0$ . That will happen at 0 and multiples of  $\pi$  (or  $180^\circ$ ). The domain then is all real numbers except multiples of  $\pi$ .

What is the range?

Like the tangent, the range will be all real numbers.

The domains and ranges of the trig functions are summarized in your book in Table 6 on page 542. You need to know these. If you know the unit circle, you can figure these out.



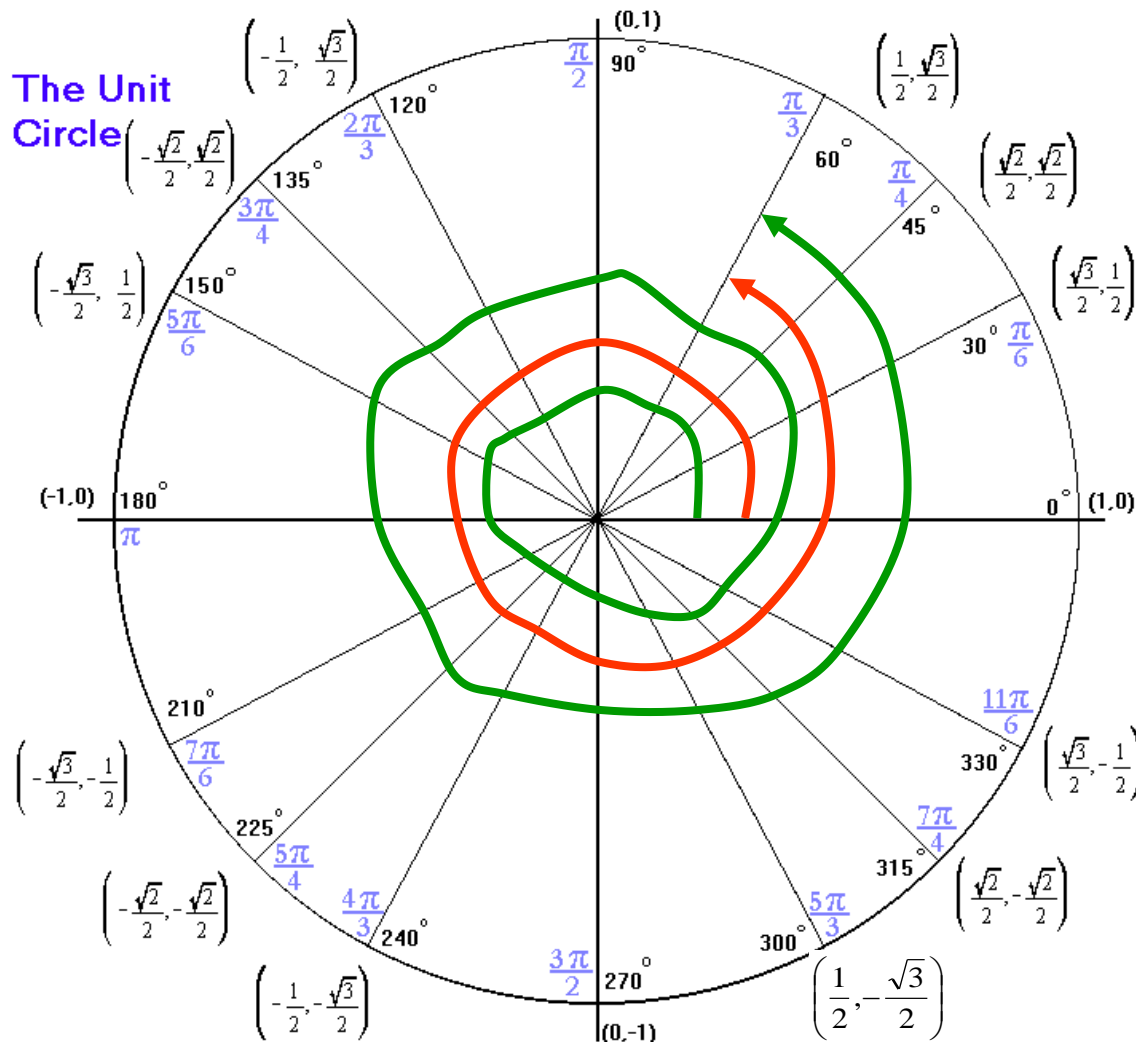


# Look at the unit circle and determine $\sin 420^\circ$ .

In fact  $\sin 780^\circ = \sin 60^\circ$  since that is just another  $360^\circ$  beyond  $420^\circ$ .

Because the sine values are equal for coterminal angles that are multiples of  $360^\circ$  added to an angle, we say that the sine is **periodic** with a period of  $360^\circ$  or  $2\pi$ .

The Unit Circle

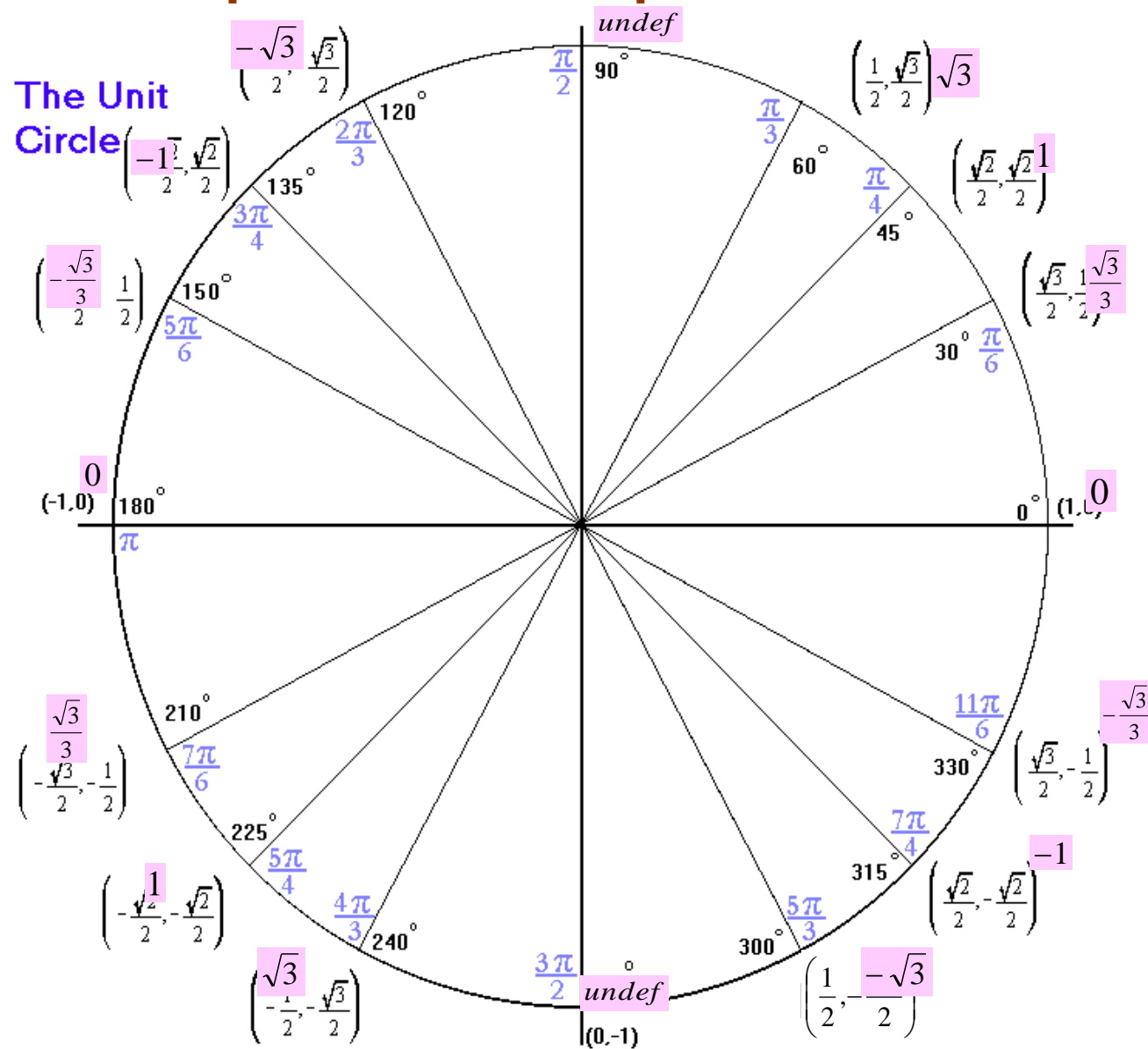


All the way around is  $360^\circ$  so we'll need more than that. We see that it will be the same as  $\sin 60^\circ$  since they are coterminal angles. So  **$\sin 420^\circ = \sin 60^\circ$** .



# The cosine is also periodic with a period of $360^\circ$ or $2\pi$ .

Let's label  
the unit  
circle with  
values of  
the tangent.  
(Remember  
this is just  
 $y/x$ )



We see that they repeat every  $\pi$  so the tangent's period is  $\pi$ .



## PERIODIC PROPERTIES

$$\sin(\theta + 2\pi) = \sin \theta$$

$$\operatorname{cosec}(\theta + 2\pi) = \operatorname{cosec} \theta$$

$$\cos(\theta + 2\pi) = \cos \theta$$

$$\sec(\theta + 2\pi) = \sec \theta$$

$$\tan(\theta + \pi) = \tan \theta$$

$$\cot(\theta + \pi) = \cot \theta$$

$$\tan \frac{9\pi}{4} = 1$$

This would have the same value as  $\tan \frac{\pi}{4}$

(you can count around on unit circle or subtract the period twice.)



# Now let's look at the unit circle to compare trig functions of positive vs. negative angles.

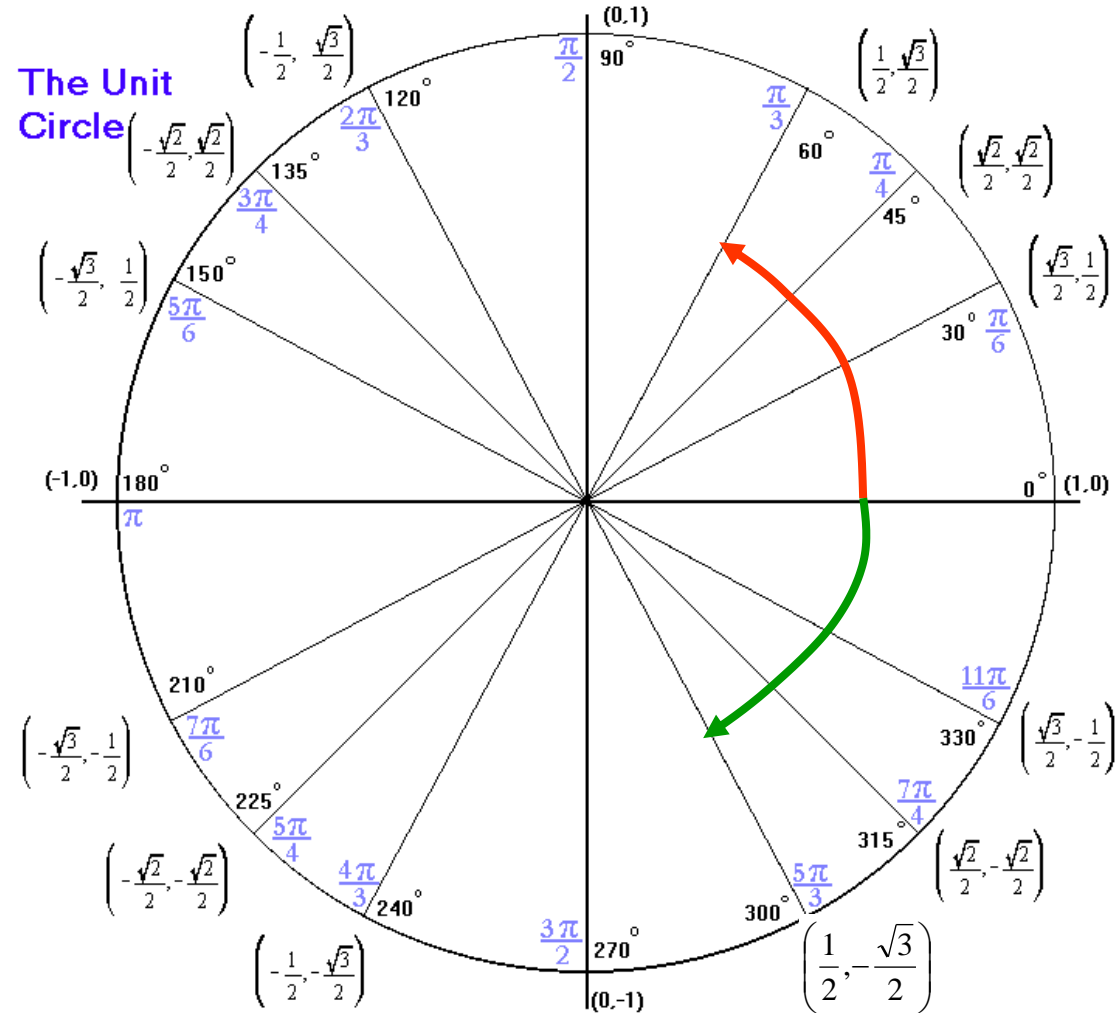
What is  $\cos \frac{\pi}{3}$ ?

$\frac{1}{2}$

What is  $\cos \left( -\frac{\pi}{3} \right)$ ?

$\frac{1}{2}$

Remember negative angle means to go clockwise





$$\cos(-\theta) = \cos\theta$$

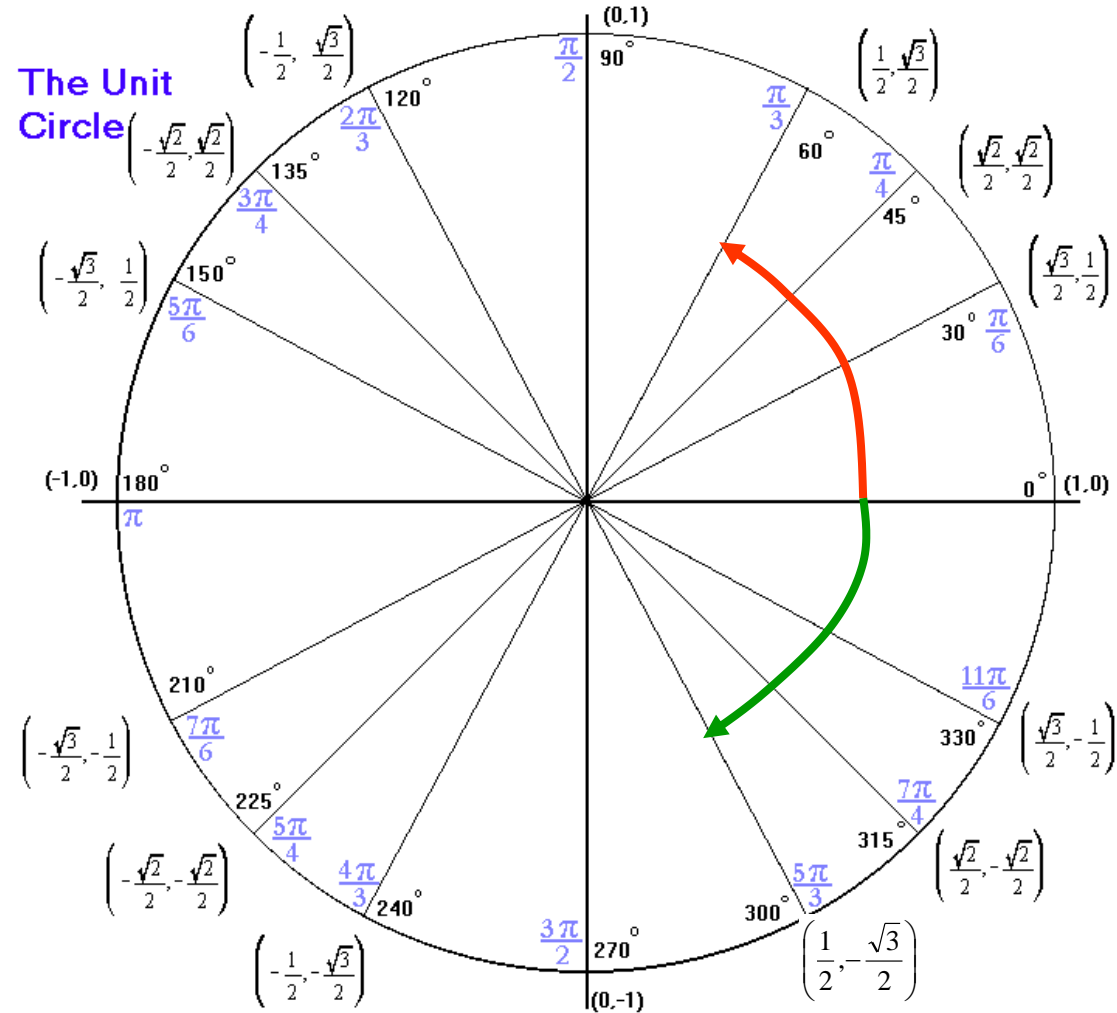
Recall from College Algebra that if we put a negative in the function and get the original back it is an even function.

What is  $\sin\frac{\pi}{3}$ ?

$$\frac{\sqrt{3}}{2}$$

What is  $\sin\left(-\frac{\pi}{3}\right)$ ?

$$-\frac{\sqrt{3}}{2}$$





$$\sin(-\theta) = -\sin\theta$$

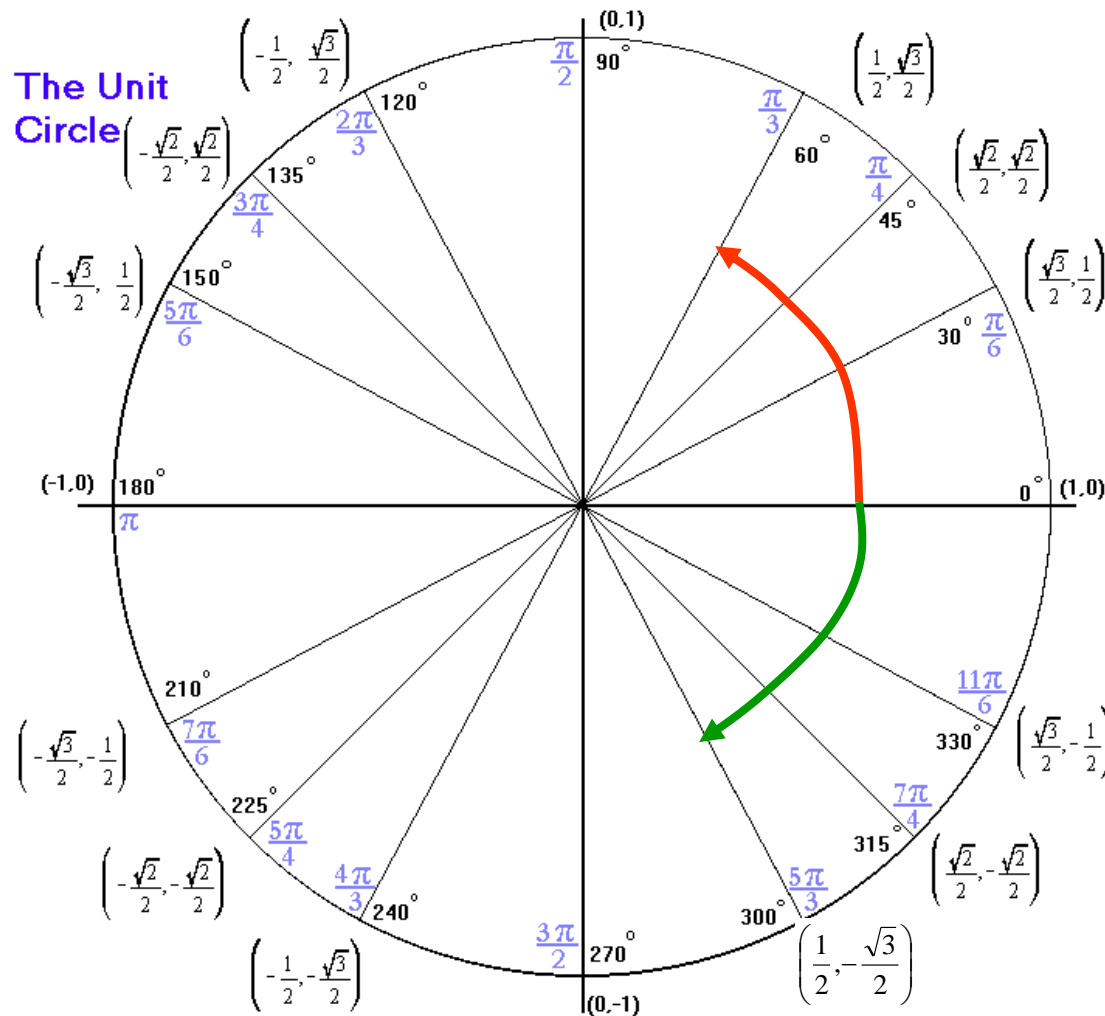
Recall from College Algebra that if we put a negative in the function and get the negative of the function back it is an odd function.

What is  $\tan \frac{\pi}{3}$ ?

$$\sqrt{3}$$

What is  $\tan\left(-\frac{\pi}{3}\right)$ ?

$$-\sqrt{3}$$





If a function is even, its reciprocal function will be also. If a function is odd its reciprocal will be also.

### EVEN-ODD PROPERTIES

$$\begin{array}{ll} \sin(-\theta) = -\sin \theta \text{ (odd)} & \operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta \text{ (odd)} \\ \cos(-\theta) = \cos \theta \text{ (even)} & \sec(-\theta) = \sec \theta \text{ (even)} \\ \tan(-\theta) = -\tan \theta \text{ (odd)} & \cot(-\theta) = -\cot \theta \text{ (odd)} \end{array}$$

$\sin(-60^\circ) =$  what in terms of a positive angle?

$$-\sin 60^\circ$$

$\sec\left(-\frac{2\pi}{3}\right) =$  what in terms of a positive angle?

$$\sec \frac{2\pi}{3}$$