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Product-to-Sum and Sum-to-Product Formulas

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Sometimes in applications you will need to re-write a product of trig functions as a sum or vice versa. We are going to look at the sum and difference formulas and derive some useful formulas to use when this situation arrives.

Let's start with the sum and difference formulas for cosine.

subtract these

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

multiply by 1/2 and write left side first

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

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Now let's use the sum and difference formulas for cosine but add this time.

add these

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta)+\cos(\alpha+\beta)=2\cos\alpha\cos\beta$$

multiply by 1/2 and write left side first

$$\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

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Now let's use the sum and difference formulas for sine.

add these

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$$

multiply by 1/2 and write left side first

$$\sin\alpha\cos\beta=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

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So here are the three product-to-sum formulas:

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha - \beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha - \beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$$

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Now we'll use these to generate some Sum-to-Product Formulas

Let's start with: $2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

Use the product-to-sum formula on this

$$\sin\begin{pmatrix}\text{1st}\\ \text{angle}\end{pmatrix}\cos\begin{pmatrix}\text{2nd}\\ \text{angle}\end{pmatrix}=\frac{1}{2}\left[\sin\left(\begin{pmatrix}\text{1st}\\ \text{angle}\end{pmatrix}+\begin{pmatrix}\text{2nd}\\ \text{angle}\end{pmatrix}\right)+\sin\left(\begin{pmatrix}\text{1st}\\ \text{angle}\end{pmatrix}-\begin{pmatrix}\text{2nd}\\ \text{angle}\end{pmatrix}\right)\right]$$

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=\sin\alpha+\sin\beta$$

$$=2\left[\frac{1}{2}\left[\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)\right]\right]$$

$$=\left[\sin\left(\frac{2\alpha}{2}\right)+\sin\left(\frac{2\beta}{2}\right)\right]=\sin\alpha+\sin\beta$$

Here are the formulas:

Combined

$$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha \pm \beta}{2}\cos\frac{\alpha \mp \beta}{2}$$

$$\cos\alpha \pm \cos\beta = \pm 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$$

Sum and Difference Formulas

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Sum and Difference Formulas for Cosines

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

(1) Find the exact value of $\cos(105^\circ)$

$$\cos(105^\circ) = \cos(60^\circ + 45^\circ) =$$

$$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ =$$

$$= \frac{1}{2} * \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} * \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} =$$

$$= \frac{\sqrt{2} - \sqrt{6}}{2}$$

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Theorem Sum and Difference Formulas for Sines

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

(2) Find the exact value of $\sin(-\frac{\pi}{12})$

$$\sin(-\frac{\pi}{12}) = \sin(\frac{\pi}{4} - \frac{\pi}{3}) =$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\sin\frac{\pi}{3} =$$

$$= \frac{\sqrt{2}}{2} * \frac{1}{2} - \frac{\sqrt{2}}{2} * \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} =$$

$$= \frac{\sqrt{2} - \sqrt{6}}{2}$$

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It is known that $\sin\alpha = \frac{1}{2}, \pi/2 < \alpha < \pi$;

and $\cos\beta = \frac{-1}{3}, \pi/2 < \beta < \pi$, find the exact value of

(a) $\cos\alpha$ (b) $\sin\beta$ (c) $\cos(\alpha - \beta)$

Theorem Sum and Difference Formulas for Tangents

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

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(9) Find the exact value $\tan(\sin^{-1}\frac{12}{13} - \cos^{-1}\frac{-3}{5})$

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Cofunction Identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$