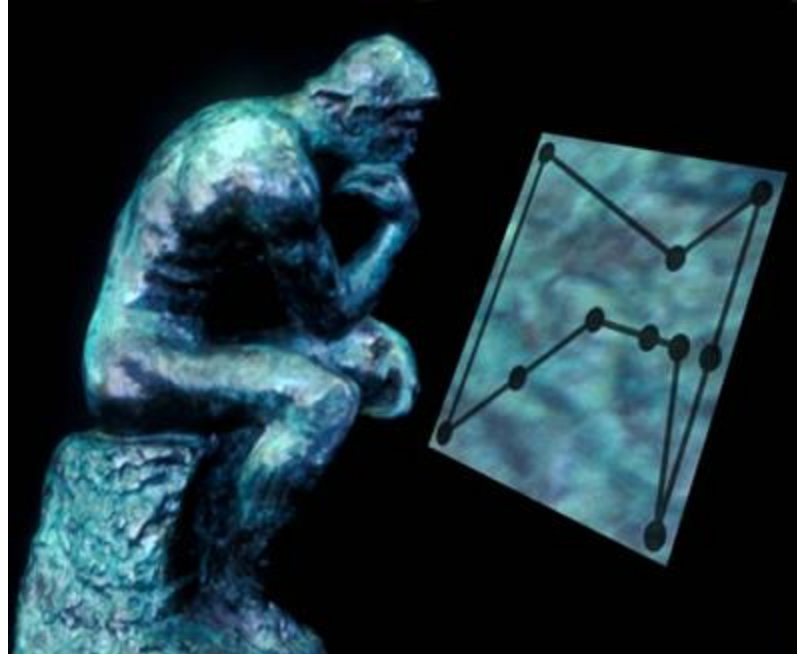


# Linear Programming





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# What is Linear Programming?





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# What is Linear Programming?

Say you own a 500 square acre farm. On this farm you can grow wheat, barley, corn or some combination of the 3. You have a limited supply of fertilizer and pesticide, both of which are needed (in different quantities) for each crop grown. Let's say wheat sells at \$7 a bushel, barley is \$3, and corn is \$3.50.



So, how many of each crop should you grow to maximize your profit?



# What Is Linear Programming?

A mathematical tool for **maximizing** or minimizing a quantity (usually **profit** or cost of production), subject to certain constraints.

Of all computations and decisions made by management in business, 50-90% of those involve linear programming.





# James Madison HIGH SCHOOL Brief Outline

- Short background to Linear Programming as a field of mathematics
- Mixture Problems & Feasible Regions
- Optimal Production (Skateboards & Dolls)
  - Corner Point Principle
  - Optimal Production Policy
  - Simplex Method
  - Tableaus (Transportation Problems)
  - Northwest Corner Algorithm
- Various Applications
- Discussion





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# Background on Linear Programming

- As a field of mathematics, LP is still a small child (in math years)
- Developed by Leonid Kantorovich around the time of WWII
- Further developed over following decades
- Today, easily the most commonly used field for optimization
- Economics, business management, transportation, technology, planning, production



the list goes on...



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# Mixture Problems

Problem where a limited number of resources are used to produce a combination of products in a fashion that **maximizes profit** from the sale of these products

Production of...

- Pretty much anything
- Skateboards & Dolls





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# Mixture Problems

- **Goal:** to find a feasible solution that maximizes profit
- ...what will our solution look like?
- Mixture Problems consist of...
  1. Resources
  2. Products
  3. Recipes
  4. Profit
  5. Objective





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# Mixture Problems

- An **Optimal Production Policy**, giving optimal number of each product to produce where:
  1. It is possible to produce this number of products &
  2. This policy gives yields maximum profit





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# Setting Up Mixture Problems

Example 1: Toy manufacturer can produce skateboards and dolls. Both require the precious resource of plastic, of which there are 60 units available.

Skateboards take **five units of plastic** and **make \$1 profit**. Dolls take **two units of plastic** and **make \$0.55 profit**.

What is the number of dolls and skateboards the company can produce to maximize profit?



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# Setting Up Mixture Problems

First identify components of the problem:

1. Resources – Plastic (60 available)
2. Products – Skateboards & Dolls
3. Recipes – Skateboards (5 units), Dolls (2 units)
4. Profits – Skateboards (\$1.00), Dolls \$0.55)
5. Objective – Maximize profit

Second, make a mixture chart:

|          |                          |   |        |
|----------|--------------------------|---|--------|
| Products | Skateboards<br>(x units) | 5 | \$1.00 |
|          | Dolls<br>(y units)       | 2 | \$0.55 |



# Translate Mixture Chart into Formulas

|          |                          | Resources    | Profit |
|----------|--------------------------|--------------|--------|
|          |                          | Plastic (60) |        |
| Products | Skateboards<br>(x units) | 5            | \$1.00 |
|          | Dolls<br>(y units)       | 2            | \$0.55 |

## 2 Groups of Equations:

- Objective Equation (profit equation)
- Constraints (minimum constraints, resource constraints...)

**Objective Equation** - total profit given number of units produced

$$P = 1x + 0.55y$$

**Constraints** – usually inequalities

$$5x + 2y \leq 60$$



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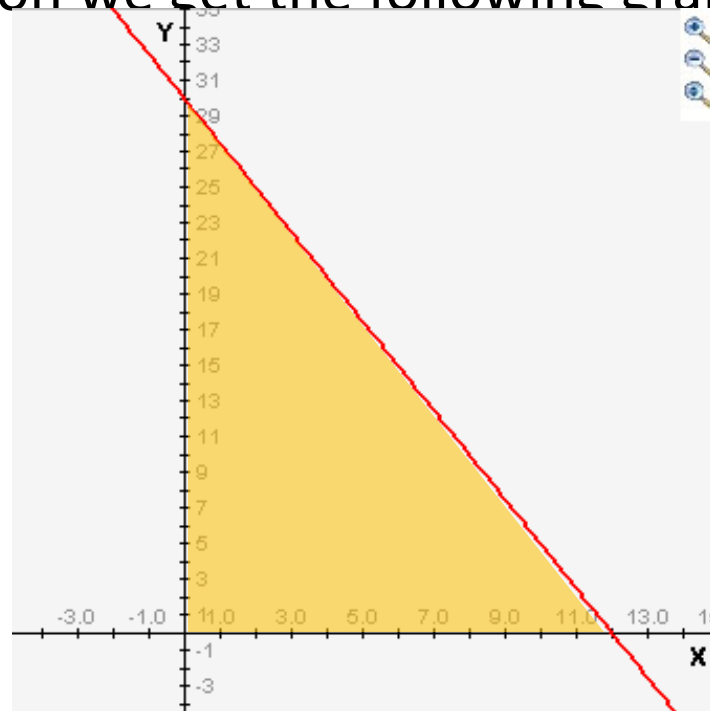
# Mixture Problems

**Feasible Region** – region which consists of all possible solution choices for a particular problem

Using the constraint equation we get the following graph:

**Constraints:**

$$5x + 2y \leq 60$$





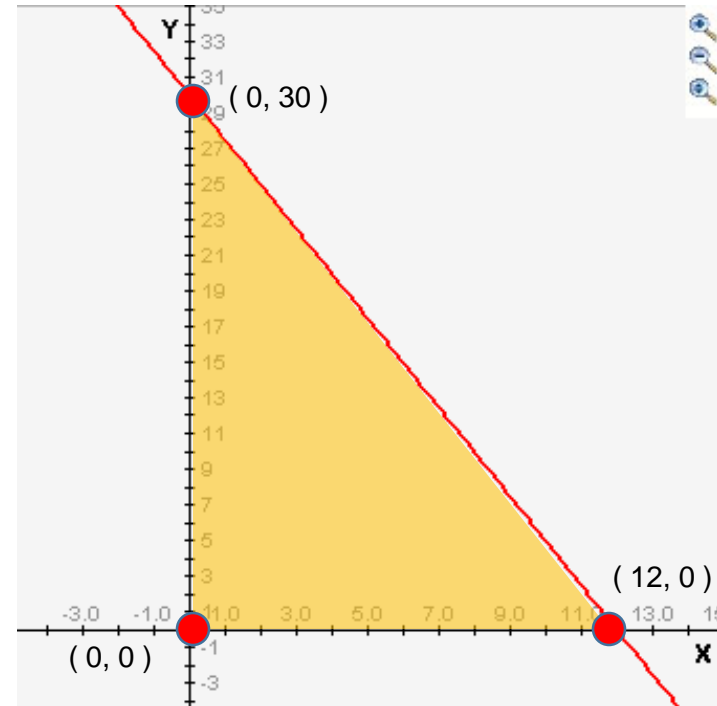
# Corner Point Principle

## Which point is optimal?

- Any point in feasible region will satisfy constraint equation, but which will maximize profit equation?

## Corner Point Principle

In LP problem, the maximal value for profit always corresponds to a corner point on feasible region





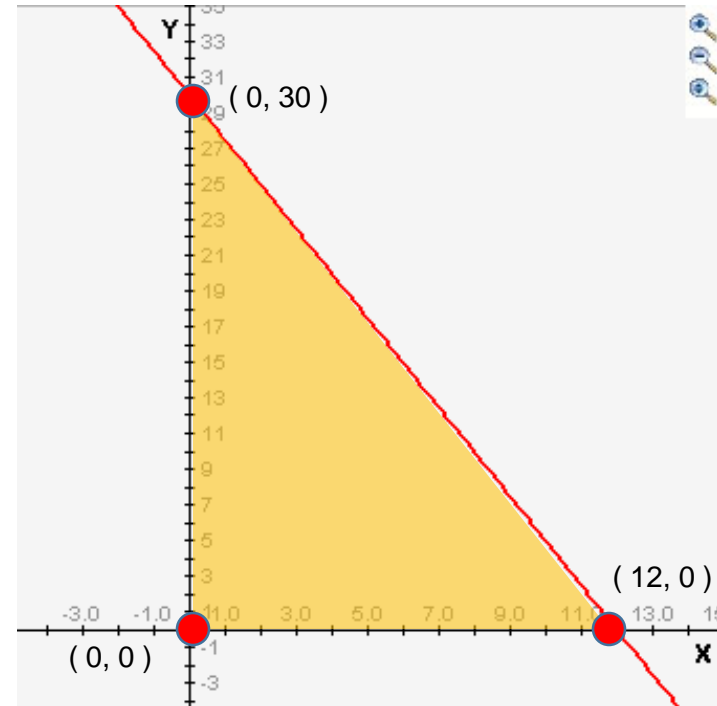
# Corner Point Principle

Plug in corner points to profit formula:

| Point          | Calculation of Profit Formula<br>$\$1.00x + \$0.55y = P$  |
|----------------|---|
| (0, 0)         | $\$1.00 (0) + \$0.55 (0) = \$0.00$                        |
| <b>(0, 30)</b> | <b><math>\\$1.00 (0) + \\$0.55 (30) = \\$16.50</math></b> |
| (12, 0)        | $\$1.00 (12) + \$0.55 (0) = \$12.00$                      |

Corner point (0,30) is the optimal point

Therefore the optimal solution would be to produce 0 skateboards and 30 dolls





# James Madison HIGH SCHOOL Quick Practice

A clothing company has 100 yards of cloth and produces shirts (x units) and vests (y units). Shirts require 10 units and have profit value of \$5, while vests require 4 units and have profit value of \$4.

What is the optimal production solution?

## Step 1 & 2:

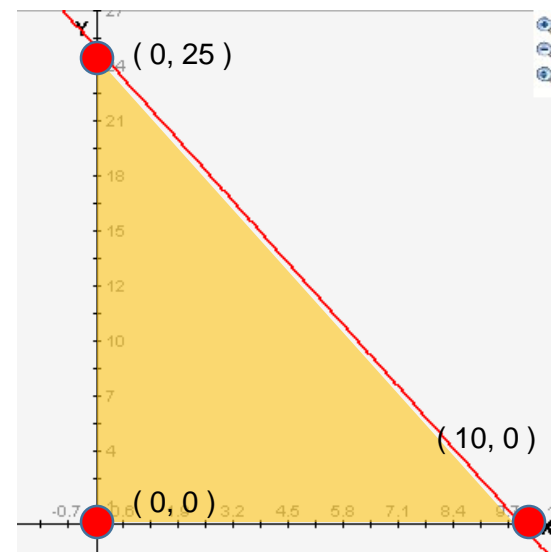
Identify Components & Mixture Chart

- Resources – Cloth (100)
- Products – Shirts & Vests
- Recipes – Shirts (10), Vests (4)
- Profits – Shirts (\$5), Vests (\$4)
- Objective – Maximize profit

|                            | Resources<br>Cloth (100) | Profit |
|----------------------------|--------------------------|--------|
| <b>Shirts</b><br>(x units) | 10                       | \$5.00 |
| <b>Vests</b><br>(y units)  | 4                        | \$4.00 |

## Steps 3 & 4:

Feasible Region & Corner Points



| Point          | Calculation of Profit Formula<br>$\$5.00x + \$4.00y = P$ |                 |
|----------------|--|-----------------|
| (0, 0)         | $\$5.00 (0) + \$4.00 (0) =$                              | \$0.00          |
| <b>(0, 25)</b> | <b><math>\\$5.00 (0) + \\$4.00 (25) =</math></b>         | <b>\$100.00</b> |
| (10, 0)        | $\$5.00 (10) + \$4.00(0) =$                              | \$50.00         |





# James Madison HIGH SCHOOL Quick Practice

What if the company decides to also put a  
“non-zero constraint” on all production?  
Must produce at least 3 shirts and 10 vests.

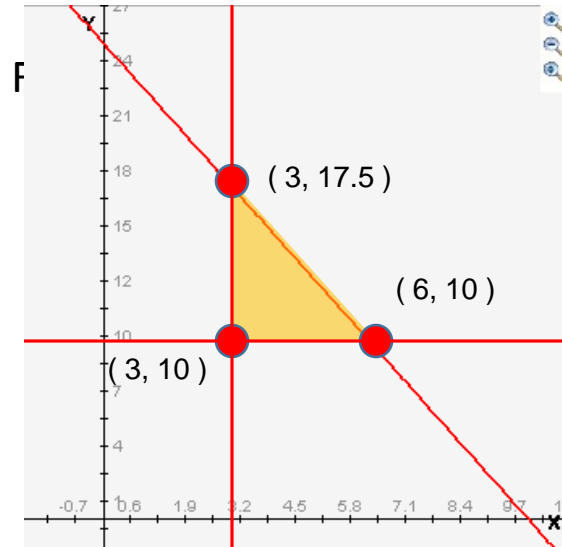
Constraints become:

$$10x + 4y \leq 100 \dots$$

$$x \geq 3$$

$$y \geq 10$$

Corner Points:



| Point            | Calculation of Profit Formula<br>$\$5.00x + \$4.00y = P$ |                |
|------------------|--|----------------|
| (3, 10)          | $\$5.00(3) + \$4.00(10) =$                               | \$55.00        |
| <b>(3, 17.5)</b> | <b><math>\\$5.00(3) + \\$4.00(17) =</math></b>           | <b>\$83.00</b> |
| (6, 10)          | $\$5.00(6) + \$4.00(10) =$                               | \$70.00        |



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# Simplex Method

- Real world problems not as simple as previous examples
- Some involve millions of “corner points” in feasible region
  - Would take fast computers days to compute.
- Simplex method – developed to help solve large, real world linear programming problems
  - Ant crawling on edges of feasible region, from corner to corner
  - Ant would do better if given temperature range (hotter, colder)
- Start with any point and evaluate profit at neighboring points. Move to neighbor with higher profit and repeat.



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# Transportation Problem

Example:

Supermarket stores get bread delivered from a bakery chain. Each store needs a certain amount per day, and the bakery only ever bakes the exact number needed. There are also shipping costs involved in delivering bread to stores.

How many breads should be shipped from each bakery  
to each store in order to minimize cost?



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# Transportation Problem

|                    |           | Store (Demand) |           |           |
|--------------------|-----------|----------------|-----------|-----------|
|                    |           | $S_1$ (3)      | $S_2$ (7) | $S_3$ (1) |
| Bakery<br>(Supply) | $B_1$ (8) | 8              | 9         | 3         |
|                    | $B_2$ (1) | 15             | 1         | 12        |
|                    | $B_3$ (2) | 1              | 3         | 5         |

The above table shows the various shipping costs from bakeries ( $B_i$ ) to stores ( $S_k$ )

To find the optimal solution we first create a **Tableau**

**Tableau** – special table showing costs and rim conditions (supply & demand) for transportation problems



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# Transportation Problem

|                         |                | Stores         |                |                |                       |
|-------------------------|----------------|----------------|----------------|----------------|-----------------------|
|                         |                | S <sub>1</sub> | S <sub>2</sub> | S <sub>3</sub> | Breads made (supply): |
| Bakeries                | B <sub>1</sub> | 8              | 9              | 3              | 8                     |
|                         | B <sub>2</sub> | 15             | 1              | 12             | 1                     |
|                         | B <sub>3</sub> | 1              | 3              | 5              | 2                     |
| Breads needed (demand): |                | 3              | 7              | 1              | 11                    |

These numbers represents The shipping cost to each store

Rim Conditions

Equal Totals from both sets of rim conditions

Number of breads sent from B<sub>3</sub> to S<sub>3</sub>

Rim Conditions



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# Transportation Problem

How to Solve the Problem using a Tableau

- Guess work...obviously not reliable, takes longer
- Northwest Corner Rule

## **Northwest Corner Rule**

1. Locate the highest and leftmost cell in table that isn't filled in. Choose the smallest rim value,  $s$ , for this cell and fill cell with it.
2. Reduce other rim value by  $s$ , and eliminate row or column that had rim value  $s$ . If more cells remain, repeat 1-2.
3. When there is a sole cell left, both its rim values will be the same, and this will be the value for the cell.



# Transportation Problem

1. Locate highest, leftmost cell and fill with smallest rim value
2. Decrement other rim value by  $s$
3. Eliminate row or column associated with  $s$

Initial Tableau:

|                         |       | Stores |       |       |                       |
|-------------------------|-------|--------|-------|-------|-----------------------|
|                         |       | $S_1$  | $S_2$ | $S_3$ | Breads made (supply): |
| Bakeries                | $B_1$ | 3      | 9     | 3     | 8                     |
|                         | $B_2$ | 15     | 1     | 12    | 1                     |
|                         | $B_3$ | 1      | 3     | 5     | 2                     |
| Breads needed (demand): |       | 3      | 7     | 1     | 11                    |



# Transportation Problem

1. Locate highest, leftmost cell and fill with smallest rim value
2. Decrement other rim value by  $s$
3. Eliminate row or column associated with  $s$
4. Repeat

New Tableau:

|                                |       | Stores |       |                              |
|--------------------------------|-------|--------|-------|------------------------------|
|                                |       | $S_2$  | $S_3$ | <b>Breads made (supply):</b> |
| Bakeries                       | $B_1$ | 9      | 3     | 5                            |
|                                | $B_2$ | 1      | 12    | 1                            |
|                                | $B_3$ | 3      | 5     | 2                            |
| <b>Breads needed (demand):</b> |       | 7      | 1     | 11                           |





# Transportation Problem

1. Locate highest, leftmost cell and fill with smallest rim value
2. Decrement other rim value by  $s$
3. Eliminate row or column associated with  $s$
4. Repeat

Tableau:

|          |                         | Stores         |                  |                       |
|----------|-------------------------|----------------|------------------|-----------------------|
|          |                         | $S_2$          | $S_3$            | Breads made (supply): |
| Bakeries | $B_1$                   | 5 <sup>9</sup> | 3 <sup>3</sup>   | 5                     |
|          | $B_2$                   | 1 <sup>1</sup> | 12 <sup>12</sup> | 1                     |
|          | $B_3$                   | 3 <sup>3</sup> | 5 <sup>5</sup>   | 2                     |
|          | Breads needed (demand): | 2 <sup>2</sup> | 1                | 11                    |



# Transportation Problem

1. Locate highest, leftmost cell and fill with smallest rim value
2. Decrement other rim value by  $s$
3. Eliminate row or column associated with  $s$
4. Repeat

Tableau:

|                         |       | Stores         |                  |                       |
|-------------------------|-------|----------------|------------------|-----------------------|
|                         |       | $S_2$          | $S_3$            | Breads made (supply): |
| Bakeries                | $B_2$ | 1 <sup>1</sup> | 12 <sup>12</sup> | 1                     |
|                         | $B_3$ | 3 <sup>3</sup> | 5 <sup>5</sup>   | 2                     |
| Breads needed (demand): |       | 2 <sup>2</sup> | 1                | 11                    |



# Transportation Problem

1. Locate highest, leftmost cell and fill with smallest rim value
2. Decrement other rim value by  $s$
3. Eliminate row or column associated with  $s$
4. Repeat

Tableau:

|       |                         | Stores |       |                       |
|-------|-------------------------|--------|-------|-----------------------|
|       |                         | $S_2$  | $S_3$ | Breads made (supply): |
| Baker | $B_3$                   | 1      | 5     | 2                     |
|       | Breads needed (demand): | 1      | 1     | 11                    |



# Transportation Problem

1. Locate highest, leftmost cell and fill with smallest rim value
2. Decrement other rim value by  $s$
3. Eliminate row or column associated with  $s$
4. Repeat

Tableau:

|              |                                | Store<br>$s$  |                              |
|--------------|--------------------------------|---|------------------------------|
|              |                                | $S_3$   | <b>Breads made (supply):</b> |
| Baker<br>$i$ | $B_3$                          | 1 <span style="border: 1px solid black; padding: 2px;">5</span> | 1                            |
|              | <b>Breads needed (demand):</b> | 1   | 11                           |



# Transportation Problem

Final Tableau:

|                         |                | Stores          |                |                 |                       |
|-------------------------|----------------|-----------------|----------------|-----------------|-----------------------|
|                         |                | S <sub>1</sub>  | S <sub>2</sub> | S <sub>3</sub>  | Breads made (supply): |
| Bakeries                | B <sub>1</sub> | 3 <sup>8</sup>  | 5 <sup>9</sup> | 1 <sup>3</sup>  | 8                     |
|                         | B <sub>2</sub> | 1 <sup>15</sup> | 1 <sup>1</sup> | 1 <sup>12</sup> | 1                     |
|                         | B <sub>3</sub> | 1 <sup>1</sup>  | 1 <sup>3</sup> | 1 <sup>5</sup>  | 2                     |
| Breads needed (demand): |                | 3               | 7              | 1               | 11                    |



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# Problems with Northwest Corner Rule?

- Only gives a feasible solution, not an optimal solution
- Does not take shipping accounts into account
  - In long run, this will not produce the optimal solution, because these costs play an important role in optimal cost
- How to get Optimal Solution?
  - Alter the feasible solution using **indicator values**



# Improving Feasible Solution

- **Indicator Value** – for a cell (not filled in) is the cost change associated with increasing or decreasing the amounts shipped in a circuit of cells starting at this empty cell.

|          |                                | Stores  |   |   |                       |
|----------|--------------------------------|---|---|---|-----------------------|
|          |                                | S <sub>1</sub>  | S <sub>2</sub>  | S <sub>3</sub>  | Breads made (supply): |
| Bakeries | B <sub>1</sub>                 | 3 <span style="border: 1px solid black; padding: 2px;">8</span> | 5 <span style="border: 1px solid black; padding: 2px;">9</span> | <span style="border: 1px solid black; padding: 2px;">3</span>   | 8                     |
|          | B <sub>2</sub>                 | <span style="border: 1px solid black; padding: 2px;">15</span>  | 1 <span style="border: 1px solid black; padding: 2px;">1</span> | <span style="border: 1px solid black; padding: 2px;">12</span>  | 1                     |
|          | B <sub>3</sub>                 | <span style="border: 1px solid black; padding: 2px;">1</span>   | 1 <span style="border: 1px solid black; padding: 2px;">3</span> | 1 <span style="border: 1px solid black; padding: 2px;">5</span> | 2                     |
|          | <b>Breads needed (demand):</b> | 3   | 7   | 1   | 11                    |

One example would be the indicator value of (B<sub>1</sub>, S<sub>3</sub>). By increasing (B<sub>1</sub>, S<sub>3</sub>), we must decrease (B<sub>1</sub>, S<sub>2</sub>), decrease (B<sub>3</sub>, S<sub>3</sub>), and increase (B<sub>3</sub>, S<sub>2</sub>).

So the cost of this move is:  
 $+ 3 - 9 + 3 - 5 = - 8$

Want moves to have negative value (decreasing cost)



# Improving Feasible Solution

- **Indicator Value** – for a cell (not filled in) is the cost change associated with increasing or decreasing the amounts shipped in a circuit of cells starting at this empty cell.

|                                |                | Stores  |   |   |                       |
|--------------------------------|----------------|---|---|---|-----------------------|
|                                |                | S <sub>1</sub>  | S <sub>2</sub>  | S <sub>3</sub>  | Breads made (supply): |
| Bakeries                       | B <sub>1</sub> | 3 <span style="border: 1px solid black; padding: 2px;">8</span> | 4 <span style="border: 1px solid black; padding: 2px;">9</span> | 1 <span style="border: 1px solid black; padding: 2px;">3</span> | 8                     |
|                                | B <sub>2</sub> | <span style="border: 1px solid black; padding: 2px;">15</span>  | 1 <span style="border: 1px solid black; padding: 2px;">1</span> | <span style="border: 1px solid black; padding: 2px;">12</span>  | 1                     |
|                                | B <sub>3</sub> | <span style="border: 1px solid black; padding: 2px;">1</span>   | 2 <span style="border: 1px solid black; padding: 2px;">3</span> | <span style="border: 1px solid black; padding: 2px;">5</span>   | 2                     |
| <b>Breads needed (demand):</b> |                | 3   | 7   | 1   | 11                    |

One example would be the indicator value of (B<sub>1</sub>, S<sub>3</sub>). By increasing (B<sub>1</sub>, S<sub>3</sub>), we must decrease (B<sub>1</sub>, S<sub>2</sub>), decrease (B<sub>3</sub>, S<sub>3</sub>), and increase (B<sub>3</sub>, S<sub>2</sub>).

So the cost of this move is:  
 $+ 3 - 9 + 3 - 5 = - 8$

Want moves to have negative value (decreasing cost)





# Improving Feasible Solution

|          |                                | Stores         |                |                |                       |
|----------|--------------------------------|----------------|----------------|----------------|-----------------------|
|          |                                | S <sub>1</sub> | S <sub>2</sub> | S <sub>3</sub> | Breads made (supply): |
| Bakeries | B <sub>1</sub>                 | 3<br>-         | 4<br>+         | 1              | 8                     |
|          | B <sub>2</sub>                 | 15             | 1              | 12             | 1                     |
|          | B <sub>3</sub>                 | 1<br>+         | 2<br>-         | 3              | 5                     |
|          | <b>Breads needed (demand):</b> | 3              | 7              | 1              | 11                    |

Next, we can look cell (B<sub>3</sub>, S<sub>1</sub>) and its indicator value:

$$+ 1 - 3 + 9 - 8 = -1$$

A negative value, so the move should be performed.

Since the minimum of the filled numbers in the cell with a negative label is 2, then we can make this move twice



# Improving Feasible Solution

|                                |                | Stores  |   |   |                       |
|--------------------------------|----------------|---|---|---|-----------------------|
|                                |                | S <sub>1</sub>  | S <sub>2</sub>  | S <sub>3</sub>  | Breads made (supply): |
| Bakeries                       | B <sub>1</sub> | 1 <span style="border: 1px solid black; padding: 2px;">8</span> | 6 <span style="border: 1px solid black; padding: 2px;">9</span> | 1 <span style="border: 1px solid black; padding: 2px;">3</span> | 8                     |
|                                | B <sub>2</sub> | <span style="border: 1px solid black; padding: 2px;">15</span>  | 1 <span style="border: 1px solid black; padding: 2px;">1</span> | <span style="border: 1px solid black; padding: 2px;">12</span>  | 1                     |
|                                | B <sub>3</sub> | 2 <span style="border: 1px solid black; padding: 2px;">1</span> | <span style="border: 1px solid black; padding: 2px;">3</span>   | <span style="border: 1px solid black; padding: 2px;">5</span>   | 2                     |
| <b>Breads needed (demand):</b> |                | 3   | 7   | 1   | 11                    |

By examining all other indicator values for remaining empty cells, we find that they all yield a positive cost change.

This means we have found an optimal solution.



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- Aviation fuel
- Store management
- Planning airline routes
- Scheduling work crews
- Energy efficiency
- Telecommunications
- Chex Mix