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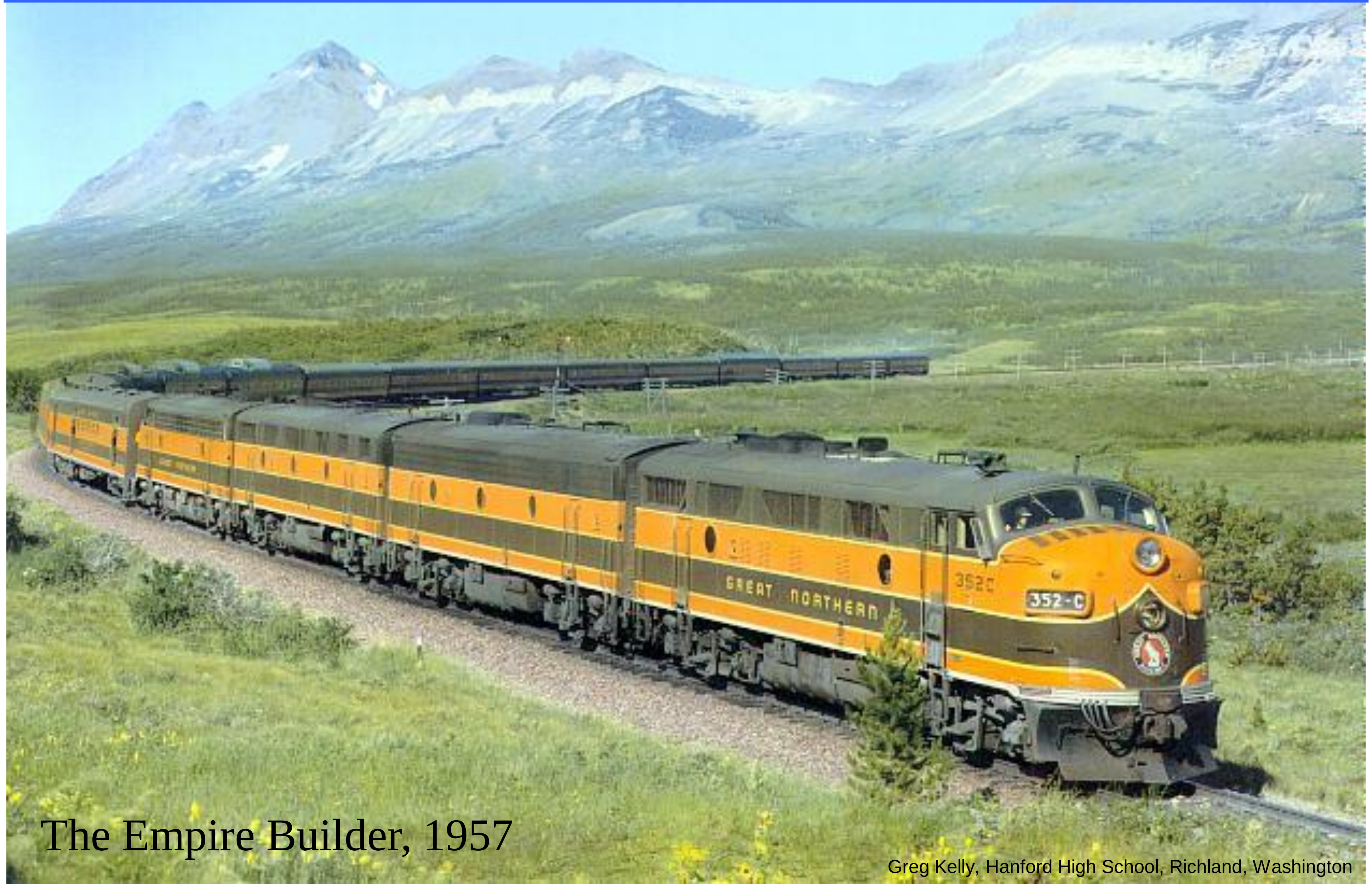
Partial Fractions



The Empire Builder, 1957

Greg Kelly, Hanford High School, Richland, Washington

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$$\int \frac{5x-3}{x^2-2x-3}\,dx$$

This would be a lot easier if we could re-write it as two separate terms.

$$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$\longleftarrow$ These are called non-repeating linear factors.

You may already know a short-cut for this type of problem. We will get to that in a few minutes.

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This would be a lot easier if we could re-write it as two separate terms.

$$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

Multiply by the common denominator.

$$5x-3 = A(x+1) + B(x-3)$$

$$5x-3 = Ax + A + Bx - B \cdot 3$$

Set like-terms equal to each other.

$$5x = Ax + Bx \qquad -3 = A - B \cdot 3$$

$$5 = A + B \qquad -3 = A - 3B$$

Solve two equations with two unknowns.

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1

$$\int \frac{5x-3}{x^2-2x-3}\,dx$$

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$$5x-3 = Ax + A + Bx - B\cdot 3$$

$$5x = Ax + Bx \qquad -3 = A - B\cdot 3$$

$$5 = A + B \qquad -3 = A - 3B$$

$$5 = A + B \qquad -3 = A - 3B$$

$$\begin{array}{r} 3 = -A + 3B \\ \hline 8 = 4B \end{array}$$

$$2 = B$$

$$5 = A + 2$$

$$3 = A$$

$$\int \frac{3}{x-3} + \frac{2}{x+1}\,dx$$

$$\boxed{3\ln|x-3| + 2\ln|x+1| + C}$$

This technique is called
Partial Fractions

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1

$$\int \frac{5x-3}{x^2-2x-3}\,dx$$

The short-cut for this type of problem is called the <u>Heaviside Method</u>, after English engineer Oliver Heaviside.

$$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

Multiply by the common denominator.

$$5x-3 = A(x+1) + B(x-3)$$

Let $x = -1$

$$-8 = \cancel{A\cdot 0} + B\cdot -4$$

$$2 = B$$

$$12 = A\cdot(4) + \cancel{B\cdot 0}$$

Let $x = 3$

$$3 = A$$

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1

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The short-cut for this type of problem is called the Heaviside Method, after English engineer Oliver Heaviside.

$$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$5x-3 = A(x+1) + B(x-3)$$

$$-8 = \cancel{A\cdot 0} + B\cdot -4$$

$$2 = B$$

$$12 = A\cdot(4) + \cancel{B\cdot 0}$$

$$3 = A$$

$$\int \frac{3}{x-3} + \frac{2}{x+1}\,dx$$

$$\boxed{3\ln|x-3| + 2\ln|x+1| + C}$$

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2

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

Repeated roots: we must use two terms for partial fractions.

$$6x+7 = A(x+2) + B$$

$$6x+7 = Ax + 2A + B$$

$$6x = Ax \qquad 7 = 2A + B$$

$$6 = A \qquad 7 = 2 \cdot 6 + B$$

$$7 = 12 + B$$

$$-5 = B$$

$$\boxed{\frac{6}{x+2} - \frac{5}{(x+2)^2}}$$

→

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3}$$

If the degree of the numerator is higher than the degree of the denominator, use long division first.

$$\begin{array}{r}
2x\\
x^2-2x-3\overline{\smash{\big)}\,2x^3-4x^2-x-3}\\
\underline{2x^3-4x^2-6x}\\
5x-3
\end{array}$$

$$2x+\frac{5x-3}{x^2-2x-3}$$

(from example one)

$$2x+\frac{5x-3}{(x-3)(x+1)}=\boxed{2x+\frac{3}{(x-3)}+\frac{2}{(x+1)}}$$

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A challenging example:

$$\frac{-2x+4}{\left(x^2+1\right)\left(x-1\right)^2}=\overbrace{\frac{Ax+B}{x^2+1}}^{\text{first degree numerator}}+\frac{C}{x-1}+\frac{D}{\left(x-1\right)^2}$$

irreducible quadratic factor

repeated root

$$-2x+4=\left(Ax+B\right)\left(x-1\right)^2+C\left(x^2+1\right)\left(x-1\right)+D\left(x^2+1\right)$$

$$-2x+4=\left(Ax+B\right)\left(x^2-2x+1\right)+C\left(x^3-x^2+x-1\right)+Dx^2+D$$

$$-2x+4=Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+Cx^3-Cx^2+Cx-C+Dx^2+D$$

→

$$-2x+4 = Ax^3 - 2Ax^2 + Ax + Bx^2 - 2Bx + B + Cx^3 - Cx^2 + Cx - C + Dx^2 + D$$

$$0 = A + C \qquad 0 = -2A + B - C + D \qquad -2 = A - 2B + C \qquad 4 = B - C + D$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 & -2 \\ 0 & 1 & -1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ +2\cdot \text{r}3 \\ -\text{r}1 \\ \\ \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -3 & 1 & 1 & -4 \\ 0 & 1 & -1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ \\ +3\cdot \text{r}2 \\ -\text{r}2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 1 & -4 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 1 & -1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ \\ \div(-2) \\ \\ \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ +\text{r}3 \end{array}$$

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$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -3 & 1 & 1 & -4 \\ 0 & 1 & -1 & 1 & 4 \end{array}\right) \begin{array}{l} \\ \\ +3\cdot r2 \\ -r2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 3 \end{array}\right) \begin{array}{l} \\ \\ \\ +r3 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right) \begin{array}{l} \\ \\ \\ \div 2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \\ \\ -r4 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} -r3 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$\rightarrow$

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$$\frac{-2x+4}{\left(x^2+1\right)\left(x-1\right)^2}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}+\frac{D}{\left(x-1\right)^2}$$

$$\boxed{=\frac{2x+1}{x^2+1}-\frac{2}{x-1}+\frac{1}{\left(x-1\right)^2}}$$

We can do this problem on the TI-89:

$$\operatorname{expand}\left(\frac{-2\cdot x+4}{\left(x^2+1\right)\cdot\left(x-1\right)^2}\right)$$

$\underbrace{\text{expand}}_{\text{F2}\ \ 3}$ ((-2x+4)/((x^2+1)*(x-1)^2))

$$\boxed{\frac{2\cdot x}{x^2+1}+\frac{1}{x^2+1}-\frac{2}{x-1}+\frac{1}{\left(x-1\right)^2}}$$

Of course with the TI-89, we could just integrate and wouldn't need partial fractions!

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