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Chapter Sections

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Solving Systems of Linear Equations by Graphing

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Systems of Linear Equations

A ***system of linear equations*** consists of two or more linear equations.

This section focuses on only two equations at a time.

The ***solution*** of a system of linear equations in two variables is any ordered pair that solves both of the linear equations.

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Solution of a System

Example

Determine whether the given point is a solution of the following system.

point: $(-3, 1)$

system: $x - y = -4$ *and* $2x + 10y = 4$

- Plug the values into the equations.

First equation: $-3 - 1 = -4$ ***true***

Second equation: $2(-3) + 10(1) = -6 + 10 = 4$ ***true***

- Since the point $(-3, 1)$ produces a true statement in both equations, it is a solution.

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Solution of a System

Example

Determine whether the given point is a solution of the following system

point: (4, 2)

system: $2x - 5y = -2$ *and* $3x + 4y = 4$

Plug the values into the equations

First equation: $2(4) - 5(2) = 8 - 10 = -2$ ***true***

Second equation: $3(4) + 4(2) = 12 + 8 = 20 \neq 4$ ***false***

Since the point (4, 2) produces a true statement in only one equation, it is NOT a solution.

Finding a Solution by Graphing

- Since our chances of guessing the right coordinates to try for a solution are not that high, we'll be more successful if we try a different technique.
- Since a solution of a system of equations is a solution common to both equations, it would also be a point common to the graphs of both equations.
- So to find the solution of a system of 2 linear equations, graph the equations and see where the lines intersect.

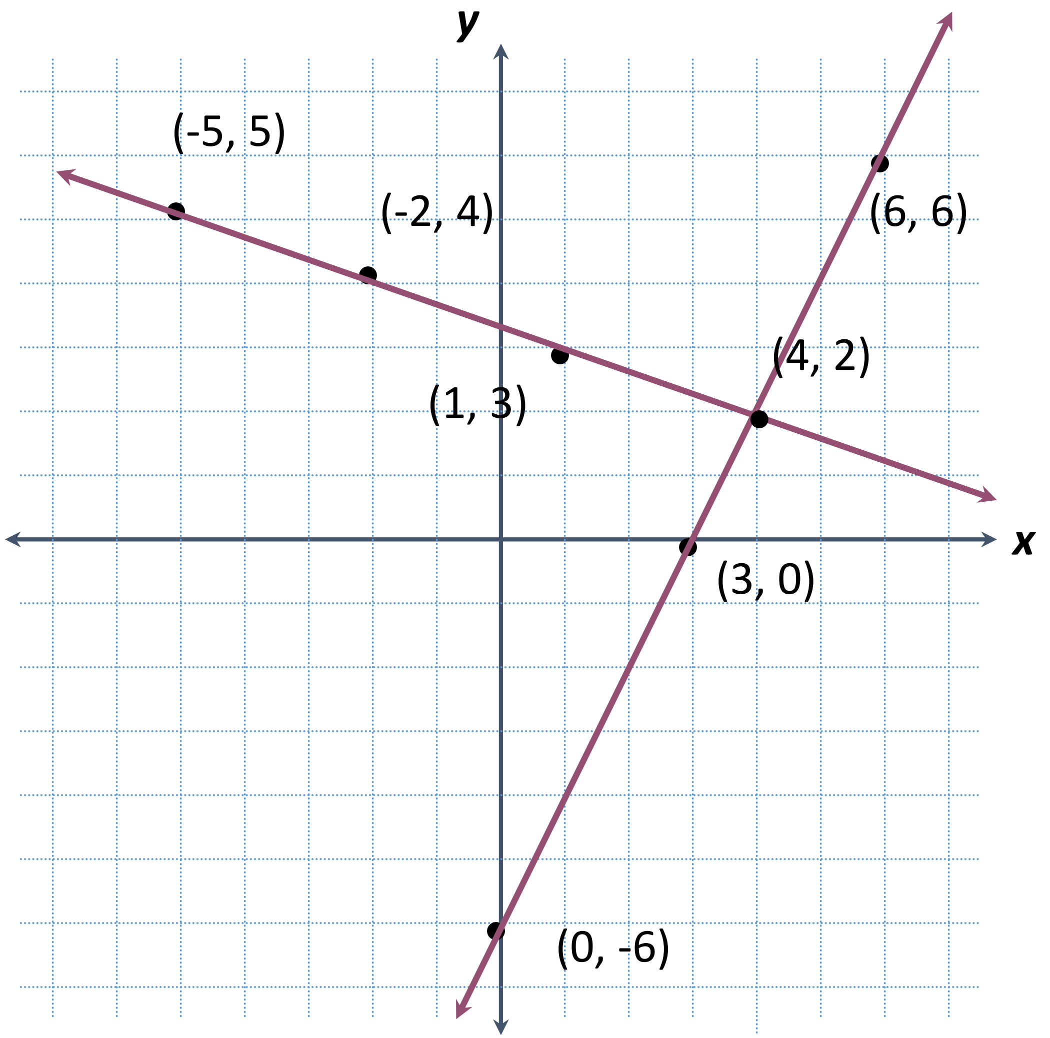
Finding a Solution by Graphing

Example

Solve the following system of equations by graphing.

$2x - y = 6$ *and*

$x + 3y = 10$



First, graph $2x - y = 6$.

Second, graph $x + 3y = 10$.

The lines APPEAR to intersect at (4, 2).

Continued.

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Finding a Solution by Graphing

Example continued

Although the solution to the system of equations appears to be (4, 2), you still need to check the answer by substituting $x = 4$ and $y = 2$ into the two equations.

First equation,

$2(4) - 2 = 8 - 2 = 6$ ***true***

Second equation,

$4 + 3(2) = 4 + 6 = 10$ ***true***

The point (4, 2) checks, so it is the solution of the system.

Finding a Solution by Graphing

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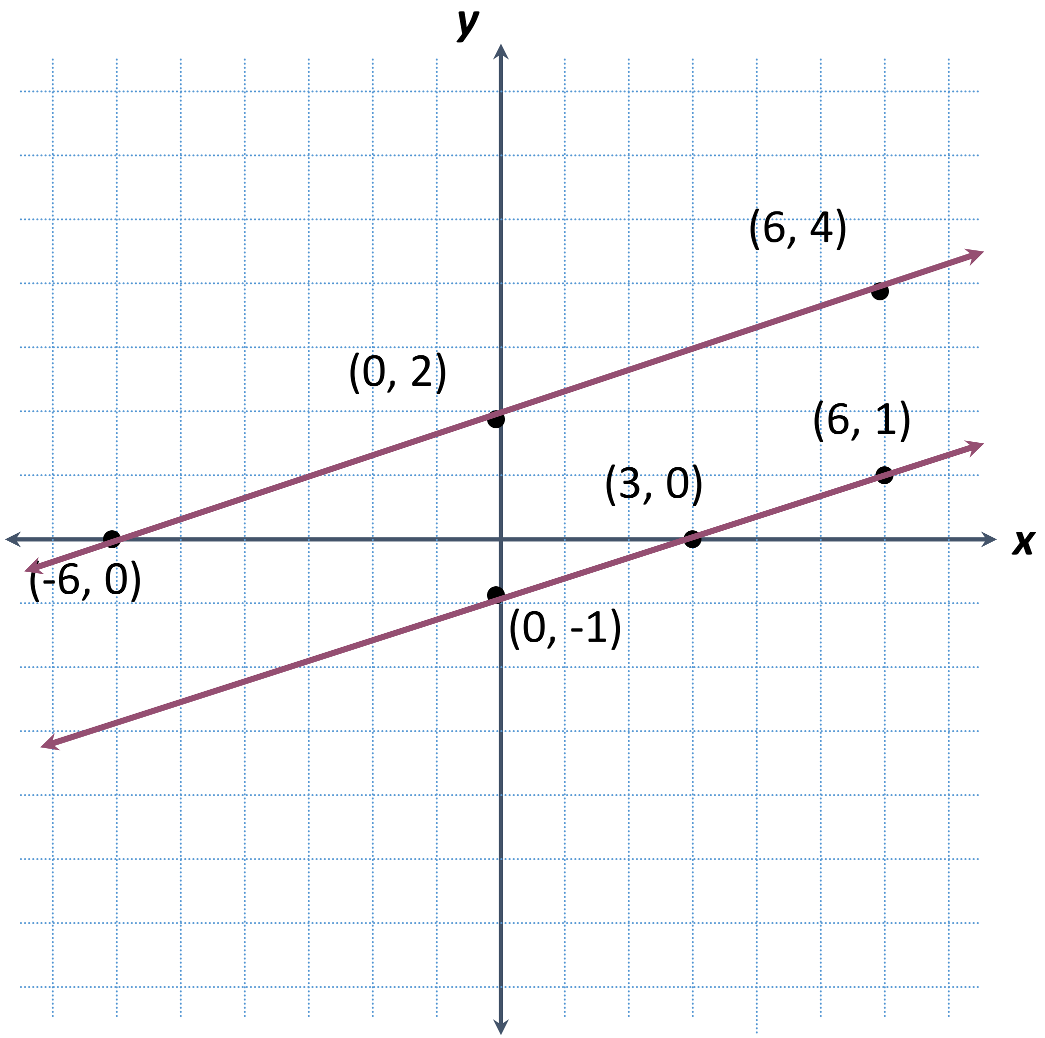
Example

Solve the following system of equations by graphing.

$-x + 3y = 6$ *and*

$3x - 9y = 9$



First, graph $-x + 3y = 6$.

Second, graph $3x - 9y = 9$.

The lines APPEAR to be parallel.

Continued.

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Finding a Solution by Graphing

Example continued

Although the lines appear to be parallel, you still need to check that they have the same slope. You can do this by solving for *y*.

First equation,

$$-x + 3y = 6$$

$$3y = x + 6 \qquad \text{(add } x \text{ to both sides)}$$

$$y = \frac{1}{3}x + 2 \qquad \text{(divide both sides by 3)}$$

Second equation,

$$3x - 9y = 9$$

$$-9y = -3x + 9 \qquad \text{(subtract } 3x \text{ from both sides)}$$

$$y = \frac{1}{3}x - 1 \qquad \text{(divide both sides by } -9\text{)}$$

Both lines have a slope of $\frac{1}{3}$, so they are parallel and do not intersect. Hence, there is no solution to the system.

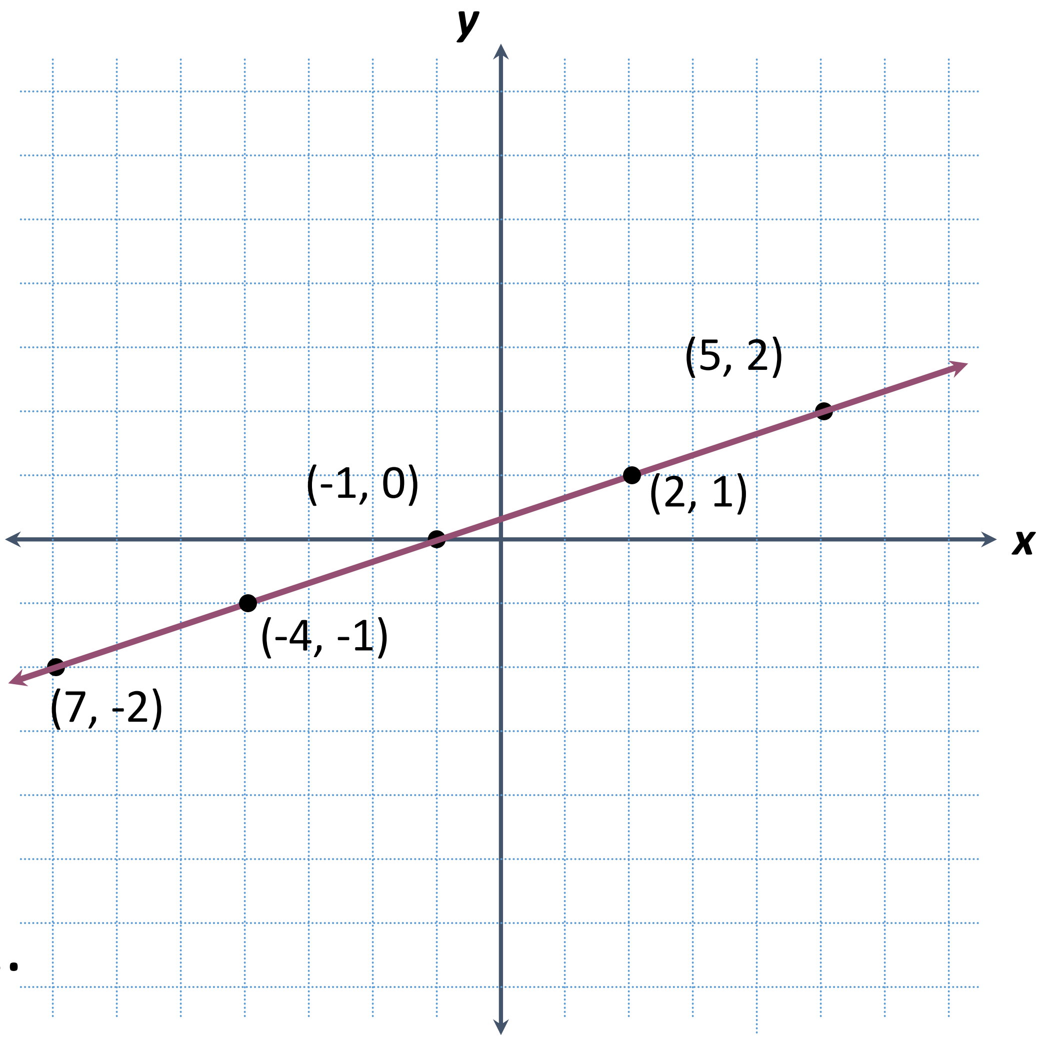
Finding a Solution by Graphing

Example

Solve the following system of equations by graphing.

$x = 3y - 1$ *and*

$2x - 6y = -2$



First, graph $x = 3y - 1$.

Second, graph $2x - 6y = -2$.

The lines APPEAR to be identical.

Continued.

Finding a Solution by Graphing

Example continued

Although the lines appear to be identical, you still need to check that they are identical equations. You can do this by solving for *y*.

First equation,

$x = 3y - 1$

$3y = x + 1$ (add 1 to both sides)

$y = \frac{1}{3}x + \frac{1}{3}$ (divide both sides by 3)

Second equation,

$2x - 6y = -2$

$-6y = -2x - 2$ (subtract $2x$ from both sides)

$y = \frac{1}{3}x + \frac{1}{3}$ (divide both sides by -6)

The two equations are identical, so the graphs must be identical. There are an infinite number of solutions to the system (all the points on the line).

Types of Systems

- There are three possible outcomes when graphing two linear equations in a plane.
  - One point of intersection, so one solution
  - Parallel lines, so no solution
  - Coincident lines, so infinite # of solutions
- If there is at least one solution, the system is considered to be ***consistent***.
- If the system defines distinct lines, the equations are ***independent***.

Types of Systems

Since there are only 3 possible outcomes with 2 lines in a plane, we can determine how many solutions of the system there will be without graphing the lines.

Change both linear equations into slope-intercept form.

We can then easily determine if the lines intersect, are parallel, or are the same line.

Types of Systems

Example

How many solutions does the following system have?

$3x + y = 1$ *and* $3x + 2y = 6$

Write each equation in slope-intercept form.

First equation,

$3x + y = 1$

$y = -3x + 1$ (subtract $3x$ from both sides)

Second equation,

$3x + 2y = 6$

$2y = -3x + 6$ (subtract $3x$ from both sides)

$y = -\frac{3}{2}x + 3$ (divide both sides by 2)

The lines are intersecting lines (since they have different slopes), so there is one solution.

Types of Systems

Example

How many solutions does the following system have?

$3x + y = 0$ *and* $2y = -6x$

Write each equation in slope-intercept form,

First equation,

$3x + y = 0$

$y = -3x$ (Subtract 3*x* from both sides)

Second equation,

$2y = -6x$

$y = -3x$ (Divide both sides by 2)

The two lines are identical, so there are infinitely many solutions.

Types of Systems

Example

How many solutions does the following system have?

$2x + y = 0$ *and* $y = -2x + 1$

Write each equation in slope-intercept form.

First equation,

$2x + y = 0$

$y = -2x$ (subtract 2x from both sides)

Second equation,

$y = -2x + 1$ (already in slope-intercept form)

The two lines are parallel lines (same slope, but different *y*-intercepts), so there are no solutions.

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Solving Systems of Linear Equations by Substitution

The Substitution Method

Another method (beside getting lucky with trial and error or graphing the equations) that can be used to solve systems of equations is called the ***substitution method***.

You solve one equation for one of the variables, then substitute the new form of the equation into the other equation for the solved variable.

The Substitution Method

Example

Solve the following system using the substitution method.

$3x - y = 6 \quad and - 4x + 2y = -8$

Solving the first equation for *y*,

$3x - y = 6$

$-y = -3x + 6$ (subtract 3*x* from both sides)

$y = 3x - 6$ (multiply both sides by – 1)

Substitute this value for *y* in the second equation.

$-4x + 2y = -8$

$-4x + 2(3x - 6) = -8$ (replace *y* with result from first equation)

$-4x + 6x - 12 = -8$ (use the distributive property)

$2x - 12 = -8$ (simplify the left side)

$2x = 4$ (add 12 to both sides)

$x = 2$ (divide both sides by 2)

Continued.

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The Substitution Method

Example continued

Substitute $x = 2$ into the first equation solved for y.

$$y = 3x - 6 = 3(2) - 6 = 6 - 6 = 0$$

Our computations have produced the point (2, 0).

Check the point in the original equations.

First equation,

$$3x - y = 6$$

$$3(2) - 0 = 6 \quad \textit{true}$$

Second equation,

$$-4x + 2y = -8$$

$$-4(2) + 2(0) = -8 \quad \textit{true}$$

The solution of the system is (2, 0).

The Substitution Method

Solving a System of Linear Equations by the Substitution Method

1) Solve one of the equations for a variable.
2) Substitute the expression from step 1 into the other equation.
3) Solve the new equation.
4) Substitute the value found in step 3 into either equation containing both variables.
5) Check the proposed solution in the original equations.

The Substitution Method

Example

Solve the following system of equations using the substitution method.

$$y = 2x - 5 \quad and \quad 8x - 4y = 20$$

Since the first equation is already solved for *y*, substitute this value into the second equation.

$$8x - 4y = 20$$

$8x - 4(2x - 5) = 20$ (replace *y* with result from first equation)

$8x - 8x + 20 = 20$ (use distributive property)

$20 = 20$ (simplify left side)

Continued.

The Substitution Method

Example continued

When you get a result, like the one on the previous slide, that is obviously true for any value of the replacements for the variables, this indicates that the two equations actually represent the same line.

There are an infinite number of solutions for this system. Any solution of one equation would automatically be a solution of the other equation.

This represents a consistent system and the linear equations are dependent equations.

The Substitution Method

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Example

Solve the following system of equations using the substitution method.

$$3x - y = 4 \quad and \quad 6x - 2y = 4$$

Solve the first equation for *y*.

$$3x - y = 4$$

$$-y = -3x + 4 \qquad \text{(subtract 3x from both sides)}$$

$$y = 3x - 4 \qquad \text{(multiply both sides by –1)}$$

Substitute this value for *y* into the second equation.

$$6x - 2y = 4$$

$$6x - 2(3x - 4) = 4 \qquad \text{(replace y with the result from the first equation)}$$

$$6x - 6x + 8 = 4 \qquad \text{(use distributive property)}$$

$$8 = 4 \qquad \text{(simplify the left side)}$$

Continued.

The Substitution Method

Example continued

When you get a result, like the one on the previous slide, that is never true for any value of the replacements for the variables, this indicates that the two equations actually are parallel and never intersect.

There is no solution to this system.

This represents an inconsistent system, even though the linear equations are independent.

Solving Systems of Linear Equations by Addition

The Elimination Method

Another method that can be used to solve systems of equations is called the ***addition*** or ***elimination method***.

You multiply both equations by numbers that will allow you to combine the two equations and eliminate one of the variables.

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The Elimination Method

Example

Solve the following system of equations using the elimination method.

$$6x - 3y = -3 \quad \textit{and} \quad 4x + 5y = -9$$

Multiply both sides of the first equation by 5 and the second equation by 3.

First equation,

$$5(6x - 3y) = 5(-3)$$

$$30x - 15y = -15$$ (use the distributive property)

Second equation,

$$3(4x + 5y) = 3(-9)$$

$$12x + 15y = -27$$ (use the distributive property)

Continued.

The Elimination Method

Example continued

Combine the two resulting equations (eliminating the variable *y*).

$$\begin{aligned} 30x - 15y &= -15 \\ \underline{12x + 15y} &\underline{= -27} \\ 42x \qquad &= -42 \\ x &= -1 \end{aligned}$$

(divide both sides by 42)

Continued.

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The Elimination Method

Example continued

Substitute the value for x into one of the original equations.

$$6x - 3y = -3$$

$$6(-1) - 3y = -3$$ (replace the x value in the first equation)

$$-6 - 3y = -3$$ (simplify the left side)

$$-3y = -3 + 6 = 3$$ (add 6 to both sides and simplify)

$$y = -1$$ (divide both sides by -3)

Our computations have produced the point $(-1, -1)$.

Continued.

The Elimination Method

Example continued

Check the point in the original equations.

First equation,

$$6x - 3y = -3$$

$$6(-1) - 3(-1) = -3 \qquad \textit{true}$$

Second equation,

$$4x + 5y = -9$$

$$4(-1) + 5(-1) = -9 \qquad \textit{true}$$

The solution of the system is $(-1, -1)$.

The Elimination Method

Solving a System of Linear Equations by the Addition or Elimination Method

1) Rewrite each equation in standard form, eliminating fraction coefficients.
2) If necessary, multiply one or both equations by a number so that the coefficients of a chosen variable are opposites.
3) Add the equations.
4) Find the value of one variable by solving equation from step 3.
5) Find the value of the second variable by substituting the value found in step 4 into either original equation.
6) Check the proposed solution in the original equations.

The Elimination Method

Example

Solve the following system of equations using the elimination method.

$$\frac{2}{3}x + \frac{1}{4}y = -\frac{3}{2}$$

$$\frac{1}{2}x - \frac{1}{4}y = -2$$

First multiply both sides of the equations by a number that will clear the fractions out of the equations.

Continued.

The Elimination Method

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Example continued

Multiply both sides of each equation by 12. (Note: you don't have to multiply each equation by the same number, but in this case it will be convenient to do so.)

First equation,

$$\frac{2}{3}x+\frac{1}{4}y=-\frac{3}{2}$$

$$12\left(\frac{2}{3}x+\frac{1}{4}y\right)=12\left(-\frac{3}{2}\right)$$ (multiply both sides by 12)

$$8x+3y=-18$$ (simplify both sides)

Continued.

The Elimination Method

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Example continued

Second equation,

$$\frac{1}{2}x - \frac{1}{4}y = -2$$

$$12\left(\frac{1}{2}x - \frac{1}{4}y\right) = 12(-2)$$ (multiply both sides by 12)

$$6x - 3y = -24$$ (simplify both sides)

Combine the two equations.

$$\begin{array}{r} 8x + 3y = -18 \\ \underline{6x - 3y = -24} \\ 14x \qquad = -42 \end{array}$$

$x = -3$ (divide both sides by 14)

Continued.

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The Elimination Method

Example continued

Substitute the value for *x* into one of the original equations.

$$8x + 3y = -18$$

$$8(-3) + 3y = -18$$

$$-24 + 3y = -18$$

$$3y = -18 + 24 = 6$$

$$y = 2$$

Our computations have produced the point (–3, 2).

Continued.

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The Elimination Method

Example continued

Check the point in the original equations. (Note: Here you should use the original equations before any modifications, to detect any computational errors that you might have made.)

First equation,

$$\frac{2}{3}x+\frac{1}{4}y=-\frac{3}{2}$$

$$\frac{2}{3}(-3)+\frac{1}{4}(2)=-\frac{3}{2}$$

$$-2+\frac{1}{2}=-\frac{3}{2}$$

true

Second equation,

$$\frac{1}{2}x-\frac{1}{4}y=-2$$

$$\frac{1}{2}(-3)-\frac{1}{4}(2)=-2$$

$$-\frac{3}{2}-\frac{1}{2}=-2$$

true

The solution is the point (–3, 2).

Special Cases

In a similar fashion to what you found in the last section, use of the addition method to combine two equations might lead you to results like . . .

5 = 5 (which is always true, thus indicating that there are infinitely many solutions, since the two equations represent the same line), or

0 = 6 (which is never true, thus indicating that there are no solutions, since the two equations represent parallel lines).

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Systems of Linear Equations and Problem Solving

Problem Solving Steps

Steps in Solving Problems

1) Understand the problem.
   - Read and reread the problem
   - Choose a variable to represent the unknown
   - Construct a drawing, whenever possible
   - Propose a solution and check
2) Translate the problem into two equations.
3) Solve the system of equations.
4) Interpret the results.
   - Check proposed solution in the problem
   - State your conclusion

Finding an Unknown Number

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Example

One number is 4 more than twice the second number. Their total is 25. Find the numbers.

1.) Understand

Read and reread the problem. Suppose that the second number is 5. Then the first number, which is 4 more than twice the second number, would have to be 14 ($4 + 2 \cdot 5$).

Is their total 25? No: $14 + 5 = 19$. Our proposed solution is incorrect, but we now have a better understanding of the problem.

Since we are looking for two numbers, we let

x = first number

y = second number

Continued

Finding an Unknown Number

Example continued

2.) *Translate*

One number is 4 more than twice the second number.

$x = 4 + 2y$

Their total is 25.

$x + y = 25$

Continued

Example continued

3.) *Solve*

We are solving the system $x = 4 + 2y$ *and* $x + y = 25$

Using substitution method, we substitute the solution for x from the first equation into the second equation.

$$x + y = 25$$

$$(4 + 2y) + y = 25$$ (replace x with result from first equation)

$$4 + 3y = 25$$ (simplify left side)

$$3y = 25 - 4 = 21$$ (subtract 4 from both sides and simplify)

$$y = 7$$ (divide both sides by 3)

Now we substitute the value for y into the first equation.

$$x = 4 + 2y = 4 + 2(7) = 4 + 14 = 18$$

Continued

Finding an Unknown Number

Example continued

4.) *Interpret*

Check: Substitute $x = 18$ and $y = 7$ into both of the equations.

First equation,

$$x = 4 + 2y$$

$$18 = 4 + 2(7) \quad \textit{true}$$

Second equation,

$$x + y = 25$$

$$18 + 7 = 25 \quad \textit{true}$$

State: The two numbers are 18 and 7.

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Solving a Problem

Example

Hilton University Drama club sold 311 tickets for a play. Student tickets cost 50 cents each; non student tickets cost \$1.50. If total receipts were \$385.50, find how many tickets of each type were sold.

1.) ***Understand***

Read and reread the problem. Suppose the number of students tickets was 200. Since the total number of tickets sold was 311, the number of non student tickets would have to be 111 (311 – 200).

Continued

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Solving a Problem

Example continued

1.) ***Understand (continued)***

Are the total receipts \$385.50? Admission for the 200 students will be 200(\$0.50), or \$100. Admission for the 111 non students will be 111(\$1.50) = \$166.50. This gives total receipts of \$100 + \$166.50 = \$266.50. Our proposed solution is incorrect, but we now have a better understanding of the problem.

Since we are looking for two numbers, we let

s = the number of student tickets

n = the number of non-student tickets

Continued

Solving a Problem

Example continued

2.) ***Translate***

Hilton University Drama club sold 311 tickets for a play.

$s + n = 311$

total receipts were $385.50

Admission for students		Admission for non students		Total receipts
$0.50s$	+	$1.50n$	=	385.50

Continued

Example continued

3.) *Solve*

We are solving the system $s + n = 311$ *and* $0.50s + 1.50n = 385.50$

Since the equations are written in standard form (and we might like to get rid of the decimals anyway), we'll solve by the addition method. Multiply the second equation by –2.

$$s + n = 311$$
$$-2(0.50s + 1.50n) = -2(385.50)$$

simplifies to

$$s + n = 311$$
$$-s - 3n = -771$$
$$-2n = -460$$
$$n = 230$$

Now we substitute the value for *n* into the first equation.

$$s + n = 311 \Rightarrow s + 230 = 311 \Rightarrow s = 81$$

Continued

Solving a Problem

Example continued

4.) *Interpret*

Check: Substitute $s = 81$ and $n = 230$ into both of the equations.

First equation,

$$s + n = 311$$

$$81 + 230 = 311$$ ***true***

Second equation,

$$0.50s + 1.50n = 385.50$$

$$0.50(81) + 1.50(230) = 385.50$$

$$40.50 + 345 = 385.50$$ ***true***

State: There were 81 student tickets and 230 non student tickets sold.

Solving a Rate Problem

Example

Terry Watkins can row about 10.6 kilometers in 1 hour downstream and 6.8 kilometers upstream in 1 hour. Find how fast he can row in still water, and find the speed of the current.

1.) *Understand*

Read and reread the problem. We are going to propose a solution, but first we need to understand the formulas we will be using. Although the basic formula is $d = r \bullet t$ (or $r \bullet t = d$), we have the effect of the water current in this problem. The rate when traveling downstream would actually be $r + w$ and the rate upstream would be $r - w$, where r is the speed of the rower in still water, and w is the speed of the water current.

Continued

Solving a Rate Problem

Example

1.) *Understand (continued)*

Suppose Terry can row 9 km/hr in still water, and the water current is 2 km/hr. Since he rows for 1 hour in each direction, downstream would be $(r + w)t = d$ or $(9 + 2)1 = 11$ km

Upstream would be $(r – w)t = d$ or $(9 – 2)1 = 7$ km

Our proposed solution is incorrect (hey, we were pretty close for a guess out of the blue), but we now have a better understanding of the problem.

Since we are looking for two rates, we let

r = the rate of the rower in still water

w = the rate of the water current

Continued

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Solving a Rate Problem

Example continued

2.) ***Translate***

rate downstream		time downstream		distance downstream
$(r + w)$	•	1	=	10.6

rate upstream		time upstream		distance upstream
$(r - w)$	•	1	=	6.8

Continued

Solving a Rate Problem

Example continued

3.) *Solve*

We are solving the system $r + w = 10.6$ *and* $r - w = 6.8$

Since the equations are written in standard form, we'll solve by the addition method. Simply combine the two equations together.

$$\begin{array}{r} r + w = 10.6 \\ r - w = 6.8 \\ \hline 2r = 17.4 \\ r = 8.7 \end{array}$$

Now we substitute the value for *r* into the first equation.

$$r + w = 10.6 \quad \Rightarrow \quad 8.7 + w = 10.6 \quad \Rightarrow \quad w = 1.9$$

Continued

Solving a Rate Problem

Example continued

4.) ***Interpret***

Check: Substitute $r = 8.7$ and $w = 1.9$ into both of the equations.

First equation,

$$(r + w)1 = 10.6$$

$$(8.7 + 1.9)1 = 10.6$$ ***true***

Second equation,

$$(r - w)1 = 1.9$$

$$(8.7 - 1.9)1 = 6.8$$ ***true***

State: Terry's rate in still water is 8.7 km/hr and the rate of the water current is 1.9 km/hr.

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Solving a Mixture Problem

Example

A Candy Barrel shop manager mixes M&M’s worth $2.00 per pound with trail mix worth $1.50 per pound. Find how many pounds of each she should use to get 50 pounds of a party mix worth $1.80 per pound.

1.) *Understand*

Read and reread the problem. We are going to propose a solution, but first we need to understand the formulas we will be using. To find out the cost of any quantity of items we use the formula

price per unit • number of units = price of all units

Continued

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Solving a Mixture Problem

Example

1.) ***Understand (continued)***

Suppose the manage decides to mix 20 pounds of M&M's. Since the total mixture will be 50 pounds, we need 50 – 20 = 30 pounds of the trail mix. Substituting each portion of the mix into the formula,

M&M's $2.00 per lb • 20 lbs = $40.00

trail mix $1.50 per lb • 30 lbs = $45.00

Mixture $1.80 per lb • 50 lbs = $90.00

Continued

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Solving a Mixture Problem

Example

1.) *Understand (continued)*

Since \$40.00 + \$45.00 $\neq$ \$90.00, our proposed solution is incorrect (hey, we were pretty close again), but we now have a better understanding of the problem.

Since we are looking for two quantities, we let

x = the amount of M&M's

y = the amount of trail mix

Continued

Solving a Mixture Problem

Example continued

2.) *Translate*

Fifty pounds of party mix

$x + y = 50$

Using price per unit • number of units = price of all units

Price of M&M's		Price of trail mix		Price of mixture
$2x$	+	$1.5y$	=	$1.8(50) = 90$

Continued

Solving a Mixture Problem

Example continued

3.) *Solve*

We are solving the system $x + y = 50$ *and* $2x + 1.50y = 90$

Since the equations are written in standard form (and we might like to get rid of the decimals anyway), we'll solve by the addition method. Multiply the first equation by 3 and the second equation by –2.

$$3(x + y) = 3(50)$$
$$-2(2x + 1.50y) = -2(90)$$

simplifies to

$$3x + 3y = 150$$
$$-4x - 3y = -180$$
$$-x = -30$$
$$x = 30$$

Now we substitute the value for *x* into the first equation.

$x + y = 50 \Rightarrow 30 + y = 50 \Rightarrow y = 20$

Continued

Solving a Mixture Problem

Example continued

4.) *Interpret*

Check: Substitute $x = 30$ and $y = 20$ into both of the equations.

First equation,

$$x + y = 50$$

$$30 + 20 = 50 \quad \textit{true}$$

Second equation,

$$2x + 1.50y = 90$$

$$2(30) + 1.50(20) = 90$$

$$60 + 30 = 90 \quad \textit{true}$$

State: The store manager needs to mix 30 pounds of M&M's and 20 pounds of trail mix to get the mixture at $1.80 a pound.