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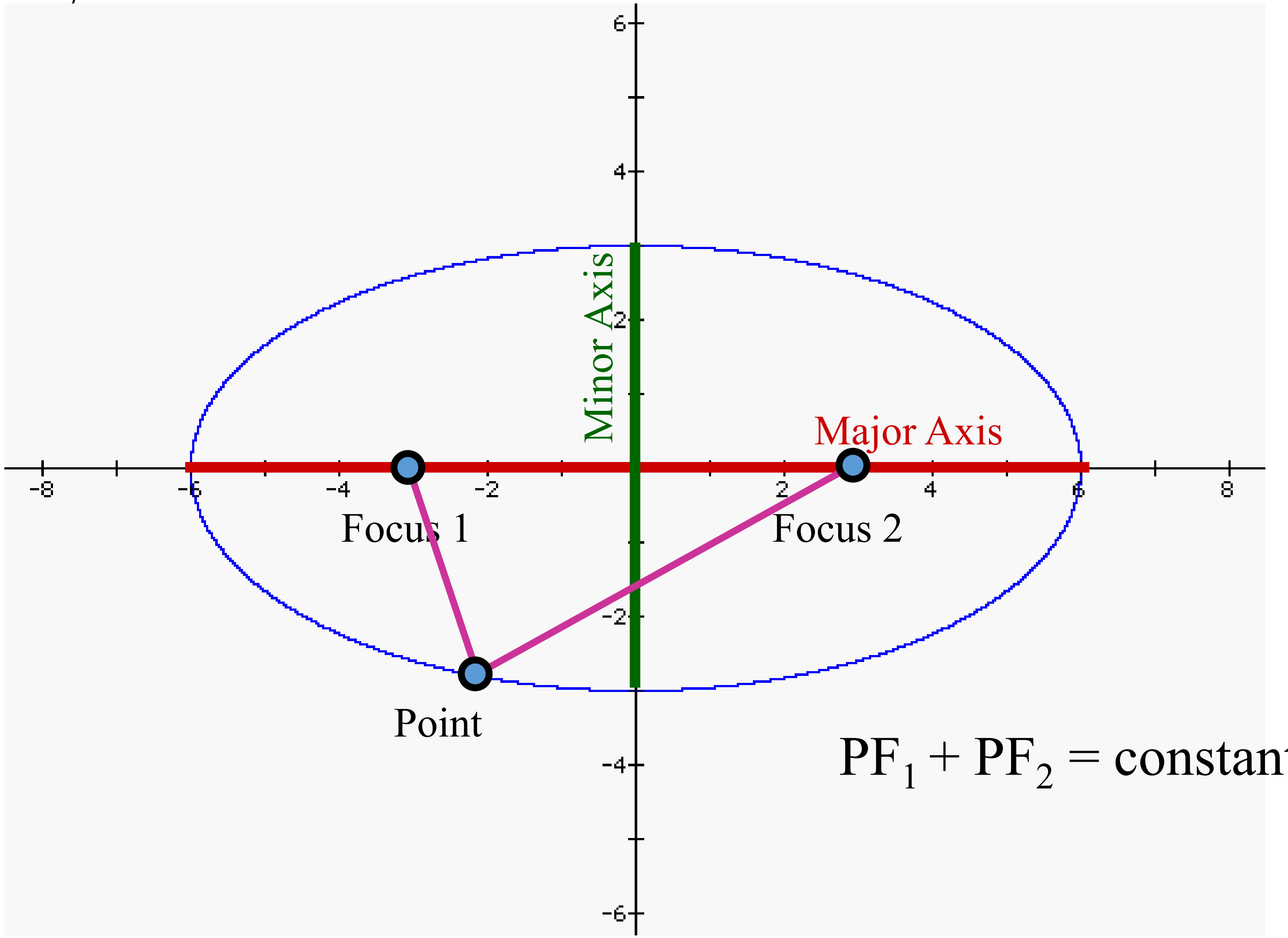
HIGH SCHOOL

# The [illegible]

# The Ellipse

An ellipse is the locus of all points in a plane such that the sum of the distances from two given points in the plane, the foci, is constant.

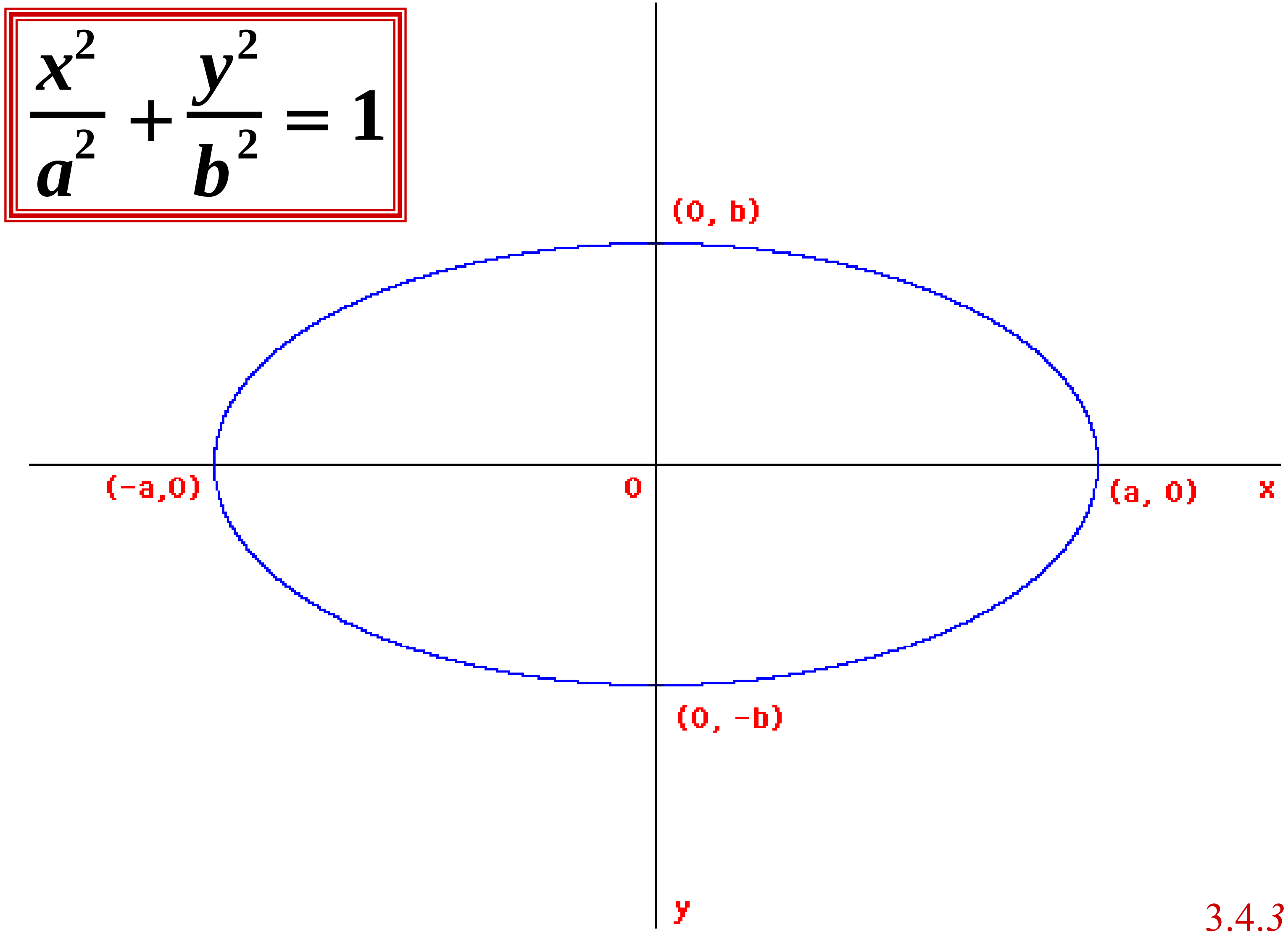



$PF_1 + PF_2 = \text{constant}$

# The Standard Forms of the Equation of the Ellipse

The standard form of an ellipse centred at the origin with the major axis of length 2*a* along the *x*-axis and a minor axis of length 2*b* along the *y*-axis, is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



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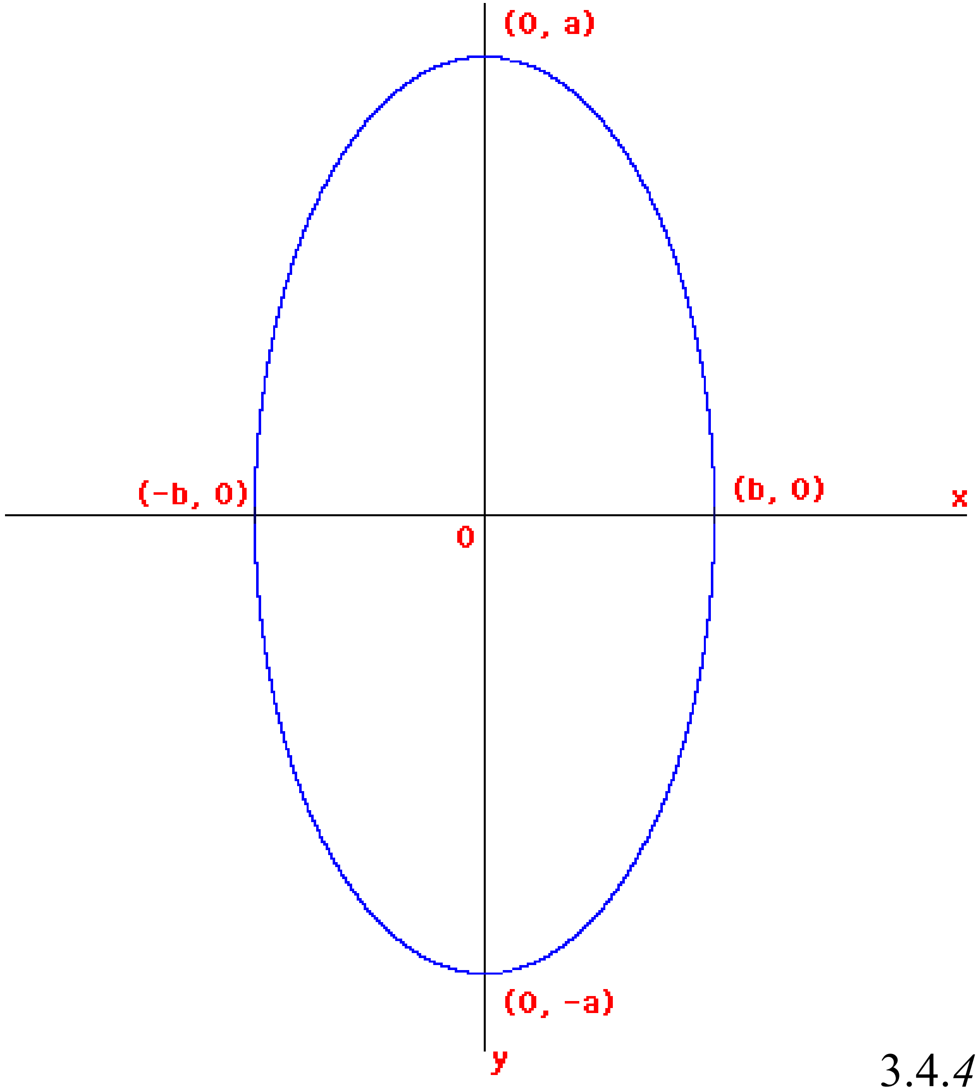
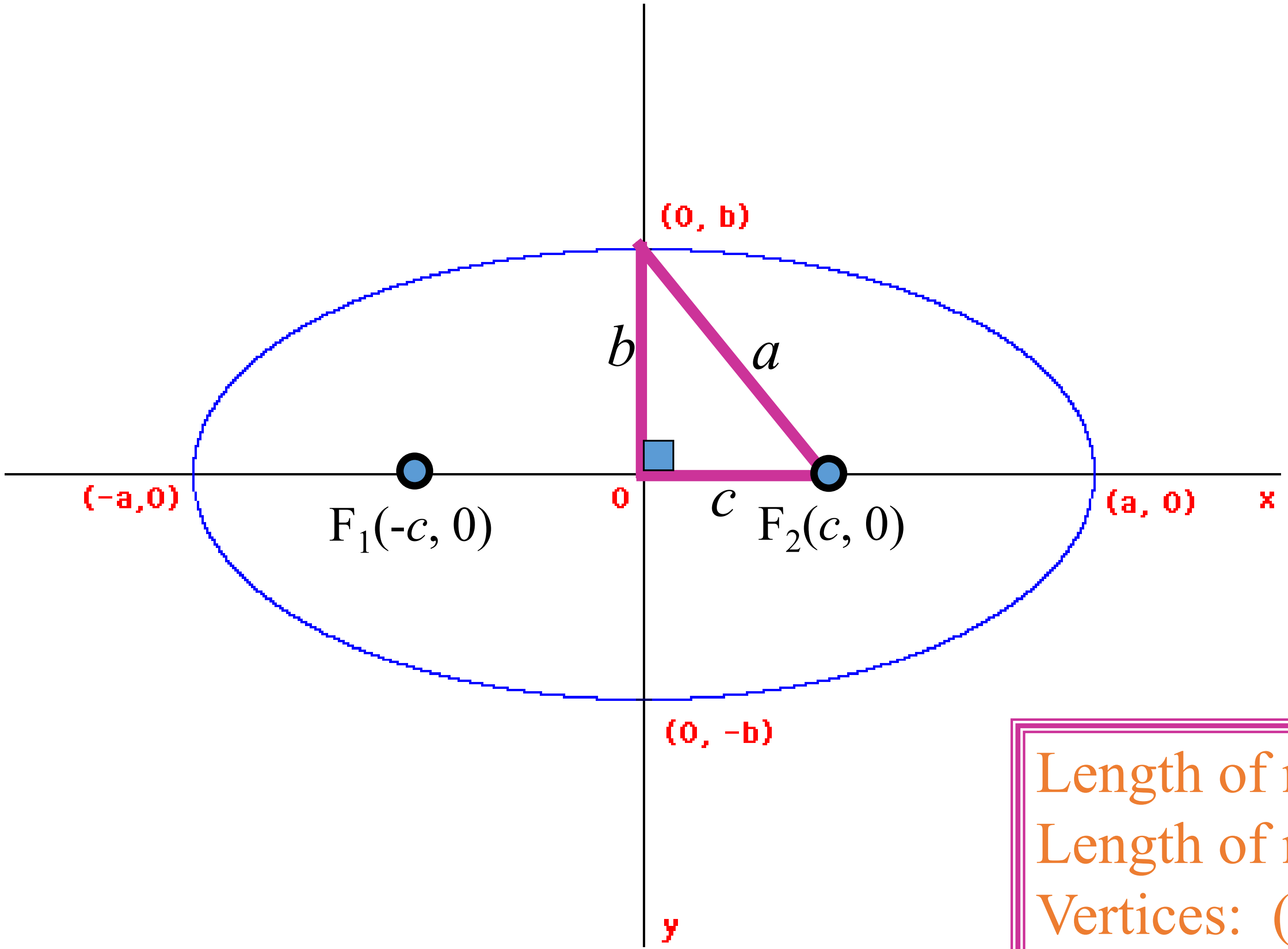
# The Standard Forms of the Equation of the Ellipse [cont'd]

The standard form of an ellipse centred at the origin with the major axis of length $2a$ along the $y$-axis and a minor axis of length $2b$ along the $x$-axis, is:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



# The Pythagorean Property




$$a^2 = b^2 + c^2$$
$$b^2 = a^2 - c^2$$
$$c^2 = a^2 - b^2$$

Length of major axis: $2a$
Length of minor axis: $2b$
Vertices: $(a, 0)$ and $(-a, 0)$
Foci: $(-c, 0)$ and $(c, 0)$

# The Standard Forms of the Equation of the Ellipse [cont'd]

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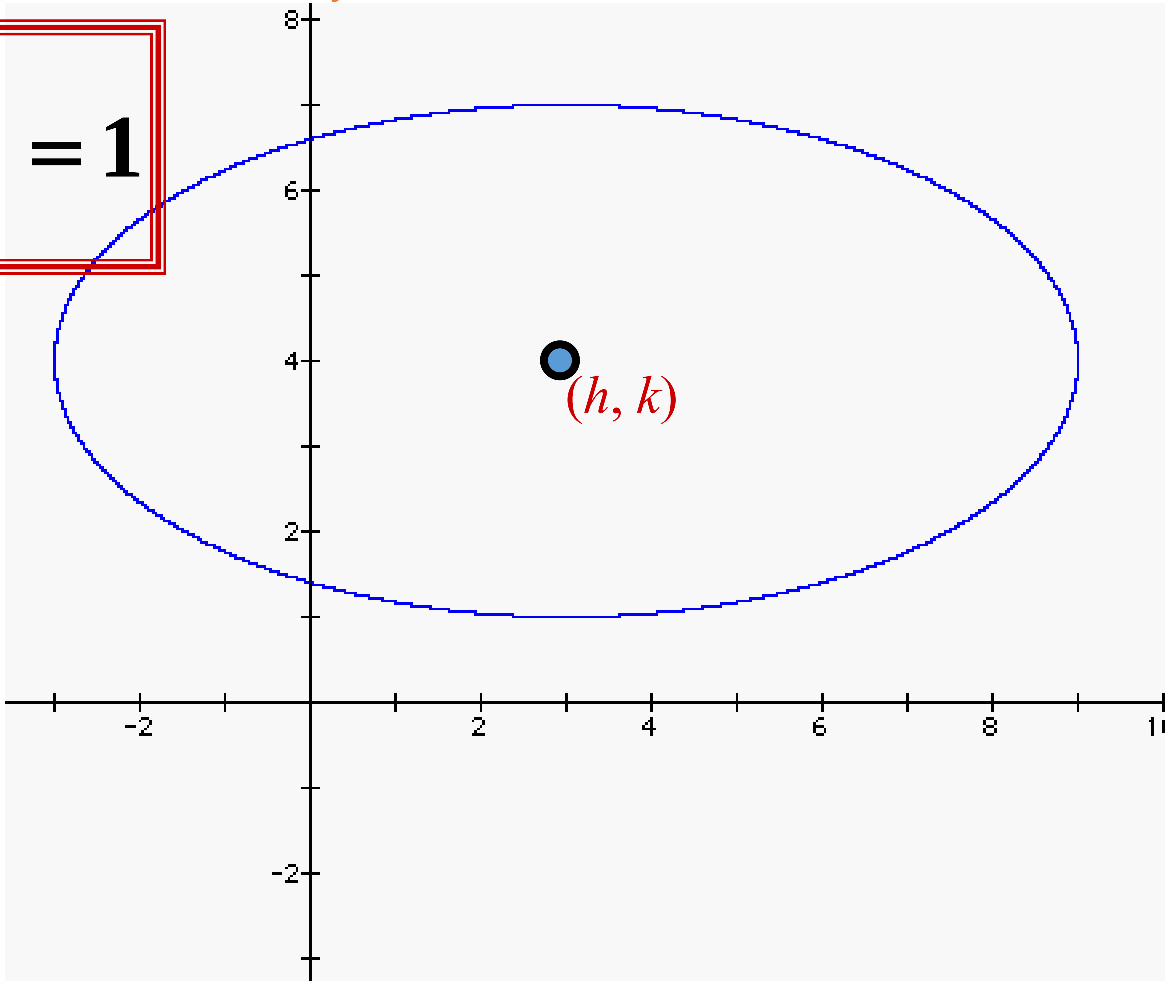
The standard form of an ellipse centred at any point $(h, k)$ with the major axis of length $2a$ parallel to the $x$-axis and a minor axis of length $2b$ parallel to the $y$-axis, is:

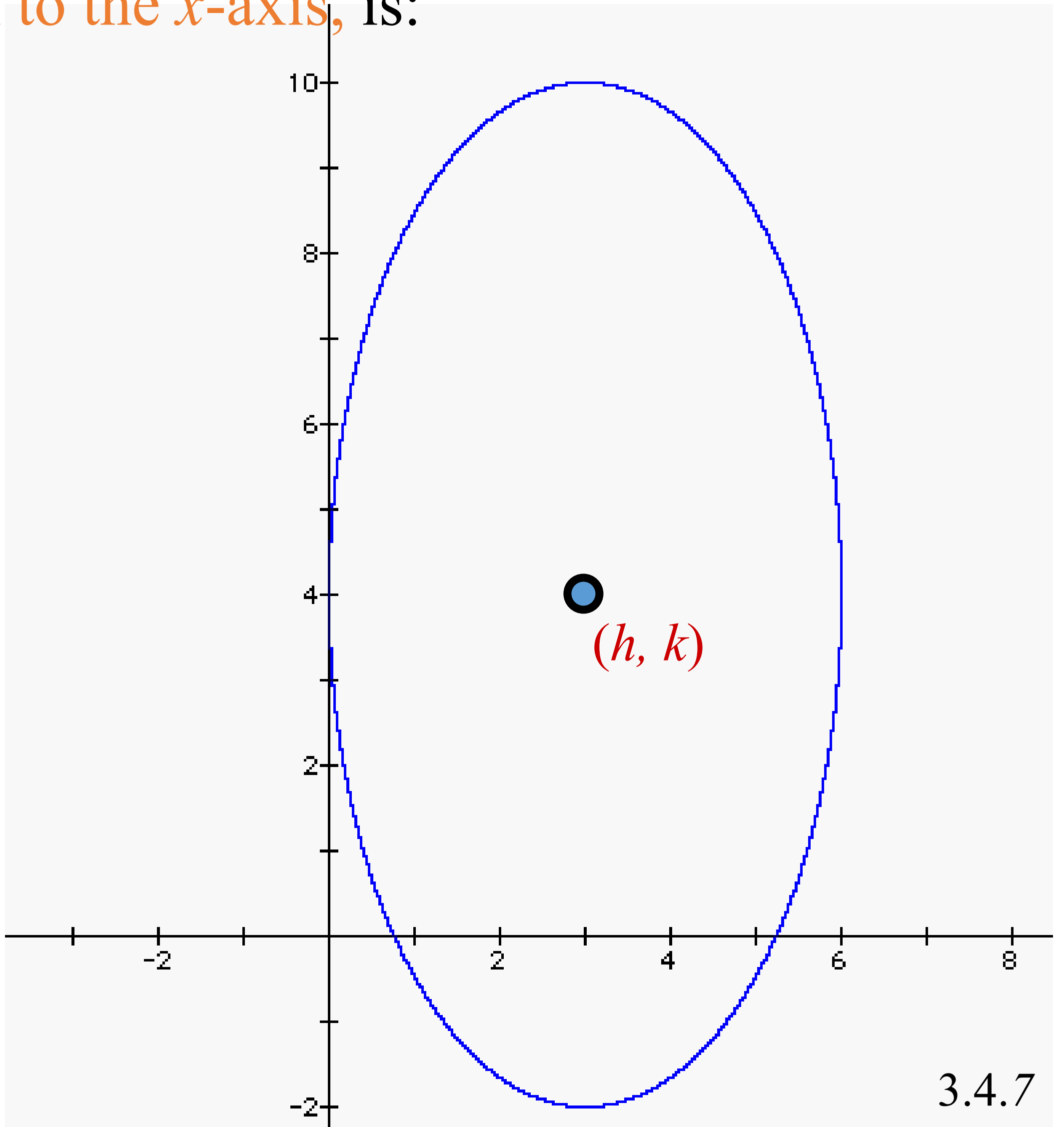
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



# The Standard Forms of the Equation of the Ellipse [cont'd]

The standard form of an ellipse centred at any point $(h, k)$ with the major axis of length $2a$ parallel to the $y$-axis and a minor axis of length $2b$ parallel to the $x$-axis, is:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$



# Finding the General Form of the Ellipse

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The general form of the ellipse is:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

$A \times C > 0$ and $A \neq C$

The general form may be found by expanding the standard form and then simplifying:

$$\frac{(x-4)^2}{3^2} + \frac{(y+2)^2}{5^2} = 1$$

$$\left[\frac{x^2 - 8x + 16}{9} + \frac{y^2 + 4y + 4}{25} = 1\right] 225$$

$$25(x^2 - 8x + 16) + 9(y^2 + 4y + 4) = 225$$

$$25x^2 - 200x + 400 + 9y^2 + 36y + 36 = 225$$

$$25x^2 + 9y^2 - 200x + 36y + 211 = 0$$

# Finding the Centre, Axes, and Foci

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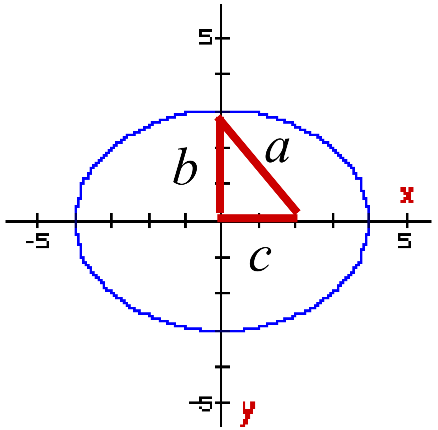
State the coordinates of the vertices, the coordinates of the foci, and the lengths of the major and minor axes of the ellipse, defined by each equation.

a) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$




The centre of the ellipse is (0, 0).

Since the larger number occurs under the $x^2$, the major axis lies on the $x$-axis.

The length of the major axis is 8.

The length of the minor axis is 6.

The coordinates of the vertices are (4, 0) and (-4, 0).

To find the coordinates of the foci, use the Pythagorean property:

$$c^2 = a^2 - b^2$$
$$= 4^2 - 3^2$$
$$= 16 - 9$$
$$= 7$$
$$c = \sqrt{7}$$

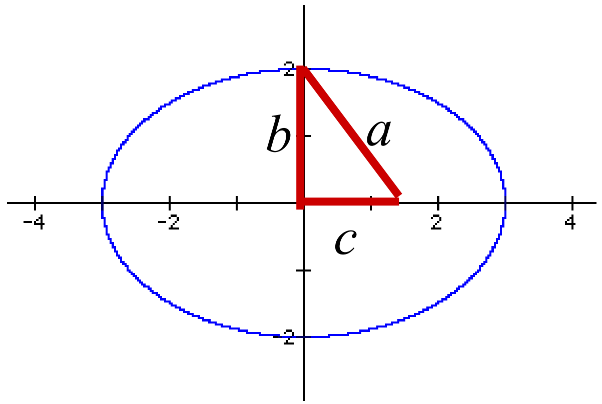
The coordinates of the foci are:

$$(-\sqrt{7}, 0) \text{ and } (\sqrt{7}, 0)$$

# Finding the Centre, Axes, and Foci

b) $4x^2 + 9y^2 = 36$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$




The centre of the ellipse is (0, 0).

Since the larger number occurs under the $x^2$, the major axis lies on the $x$-axis.

The length of the major axis is 6.

The length of the minor axis is 4.

The coordinates of the vertices are (3, 0) and (-3, 0).

To find the coordinates of the foci, use the Pythagorean property.

$$c^2 = a^2 - b^2$$
$$= 3^2 - 2^2$$
$$= 9 - 4$$
$$= 5$$
$$c = \sqrt{5}$$

The coordinates of the foci are:

$(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$

# Finding the Equation of the Ellipse With Centre at (0, 0)

a) Find the equation of the ellipse with centre at (0, 0), foci at (5, 0) and (-5, 0), a major axis of length 16 units, and a minor axis of length 8 units.

Since the foci are on the *x*-axis, the major axis is the *x*-axis.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The length of the major axis is 16 so $a = 8$.
The length of the minor axis is 8 so $b = 4$.

$$\frac{x^2}{8^2} + \frac{y^2}{4^2} = 1$$

$$\frac{x^2}{64} + \frac{y^2}{16} = 1$$

Standard form

$$64\left[\frac{x^2}{64} + \frac{y^2}{16}\right] = [1]\,64$$

$$x^2 + 4y^2 = 64$$

$$x^2 + 4y^2 - 64 = 0$$

General form

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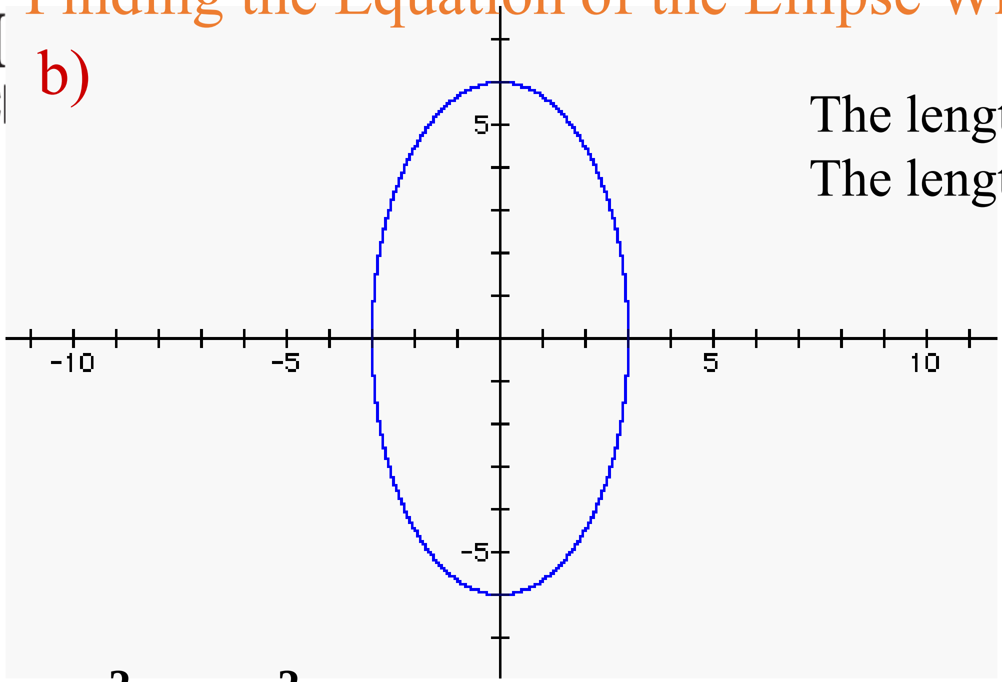
# Finding the Equation of the Ellipse With Centre at (0, 0)

b)




The length of the major axis is 12 so $a = 6$.
The length of the minor axis is 6 so $b = 3$.

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

Standard form

$$36\left[\frac{x^2}{9} + \frac{y^2}{36}\right] = [1]\,36$$

$$4x^2 + y^2 = 36$$

$$4x^2 + y^2 - 36 = 0$$

General form

# Finding the Equation of the Ellipse With Centre at $(h, k)$

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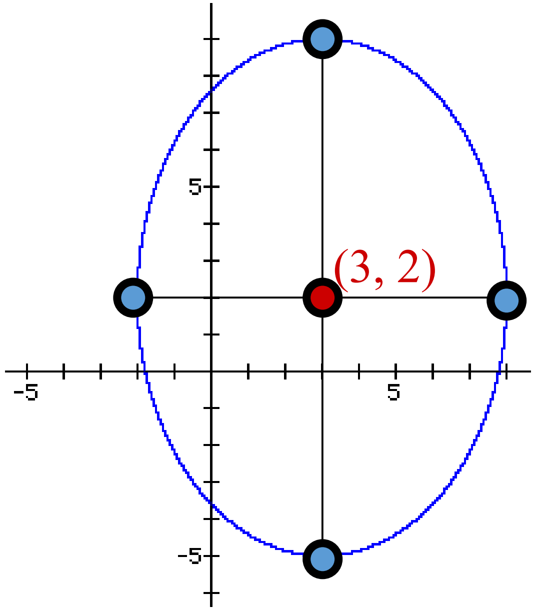
a) Find the equation for the ellipse with the centre at (3, 2), passing through the points (8, 2), (-2, 2), (3, -5), and (3, 9).

The major axis is parallel to the $y$-axis and has a length of 14 units, so $a = 7$.
The minor axis is parallel to the $x$-axis and has a length of 10 units, so $b = 5$.
The centre is at (3, 2), so $h = 3$ and $k = 2$.




$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-3)^2}{5^2} + \frac{(y-2)^2}{7^2} = 1$$

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{49} = 1$$ Standard form

$$49(x - 3)^2 + 25(y - 2)^2 = 1225$$

$$49(x^2 - 6x + 9) + 25(y^2 - 4y + 4) = 1225$$

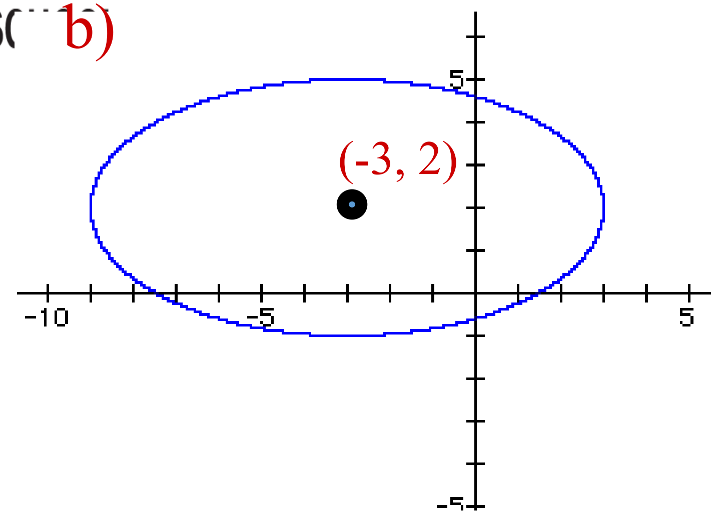
$$49x^2 - 294x + 441 + 25y^2 - 100y + 100 = 1225$$

$$49x^2 + 25y^2 - 294x - 100y + 541 = 1225$$

$$49x^2 + 25y^2 - 294x - 100y - 684 = 0$$ General form

# Finding the Equation of the Ellipse With Centre at $(h, k)$

b)




The major axis is parallel to the $x$-axis and has a length of 12 units, so $a = 6$.
The minor axis is parallel to the $y$-axis and has a length of 6 units, so $b = 3$.
The centre is at (-3, 2), so $h = -3$ and $k = 2$.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-(-3))^2}{6^2} + \frac{(y-2)^2}{3^2} = 1$$

$$\frac{(x+3)^2}{36} + \frac{(y-2)^2}{9} = 1$$

Standard form

$$(x+3)^2 + 4(y-2)^2 = 36$$

$$(x^2 + 6x + 9) + 4(y^2 - 4y + 4) = 36$$

$$x^2 + 6x + 9 + 4y^2 - 16y + 16 = 36$$

$$x^2 + 4y^2 + 6x - 16y + 25 = 36$$

$$x^2 + 4y^2 + 6x - 16y - 11 = 0$$

General form

# Analysis of the Ellipse

Find the coordinates of the centre, the length of the major and minor axes, and the coordinates of the foci of each ellipse:

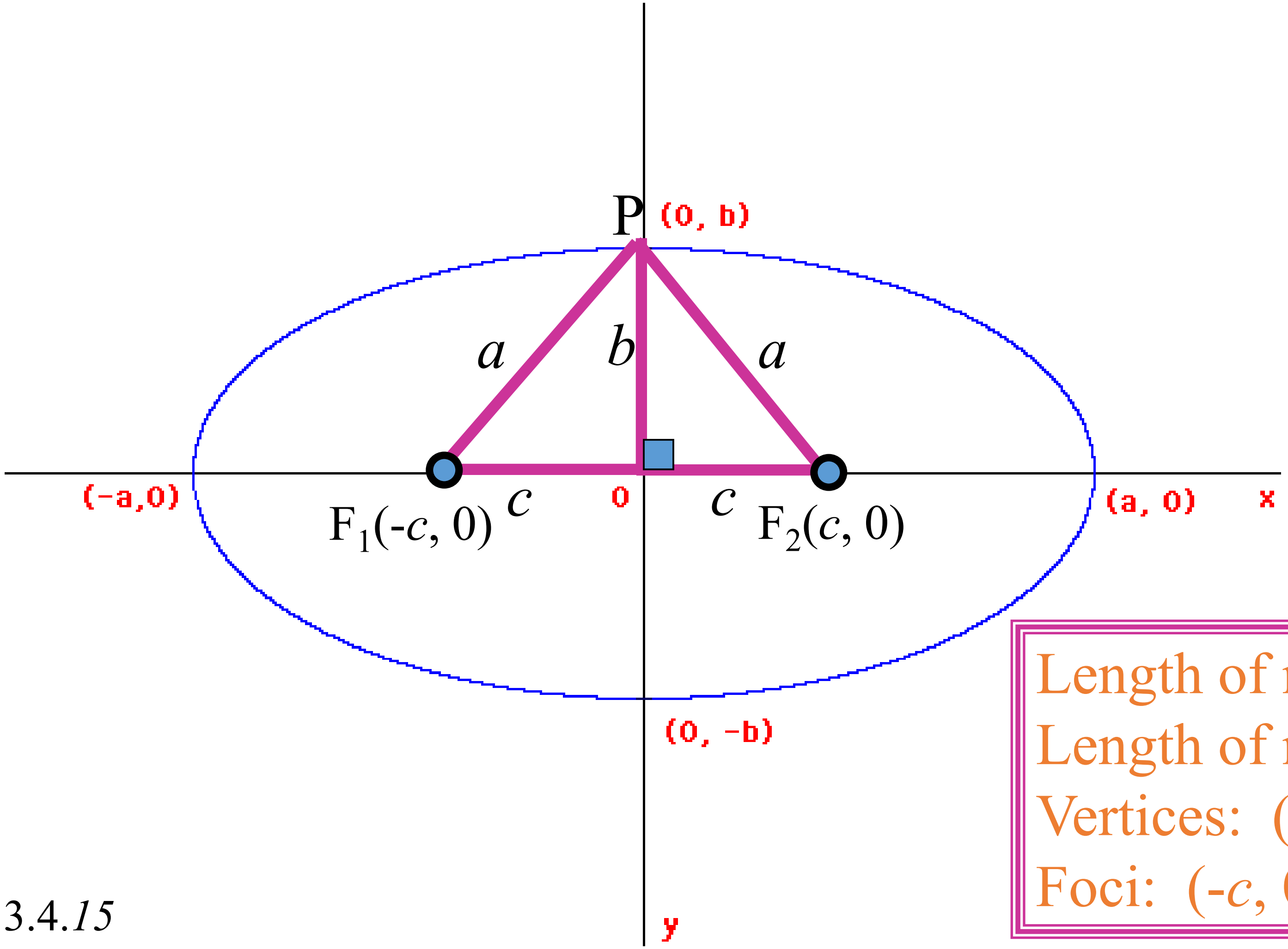
Recall:

$PF_1 + PF_2 = 2a$

$a^2 = b^2 + c^2$

$b^2 = a^2 - c^2$

$c^2 = a^2 - b^2$




Length of major axis: $2a$
Length of minor axis: $2b$
Vertices: $(a, 0)$ and $(-a, 0)$
Foci: $(-c, 0)$ and $(c, 0)$

# Analysis of the Ellipse [cont'd]

a) $x^2 + 4y^2 - 2x + 8y - 11 = 0$

$$x^2 + 4y^2 - 2x + 8y - 11 = 0$$

$$(x^2 - 2x) + (4y^2 + 8y) - 11 = 0$$

$$(x^2 - 2x + \underline{\quad 1 \quad}) + 4(y^2 + 2y + \underline{\quad 1 \quad}) = 11 + \underline{\quad 1 \quad} + \underline{\quad 4 \quad}$$

$$(x - 1)^2 + 4(y + 1)^2 = 16$$

$$\frac{(x-1)^2}{16} + \frac{(y+1)^2}{4} = 1$$

Since the larger number occurs under the $x^2$, the major axis is parallel to the $x$-axis.

$h = 1$
$k = -1$
$a = 4$
$b = 2$

$$c^2 = a^2 - b^2$$
$$= 4^2 - 2^2$$
$$= 16 - 4$$
$$= 12$$

$$c = \sqrt{12}$$

$$c = 2\sqrt{3}$$

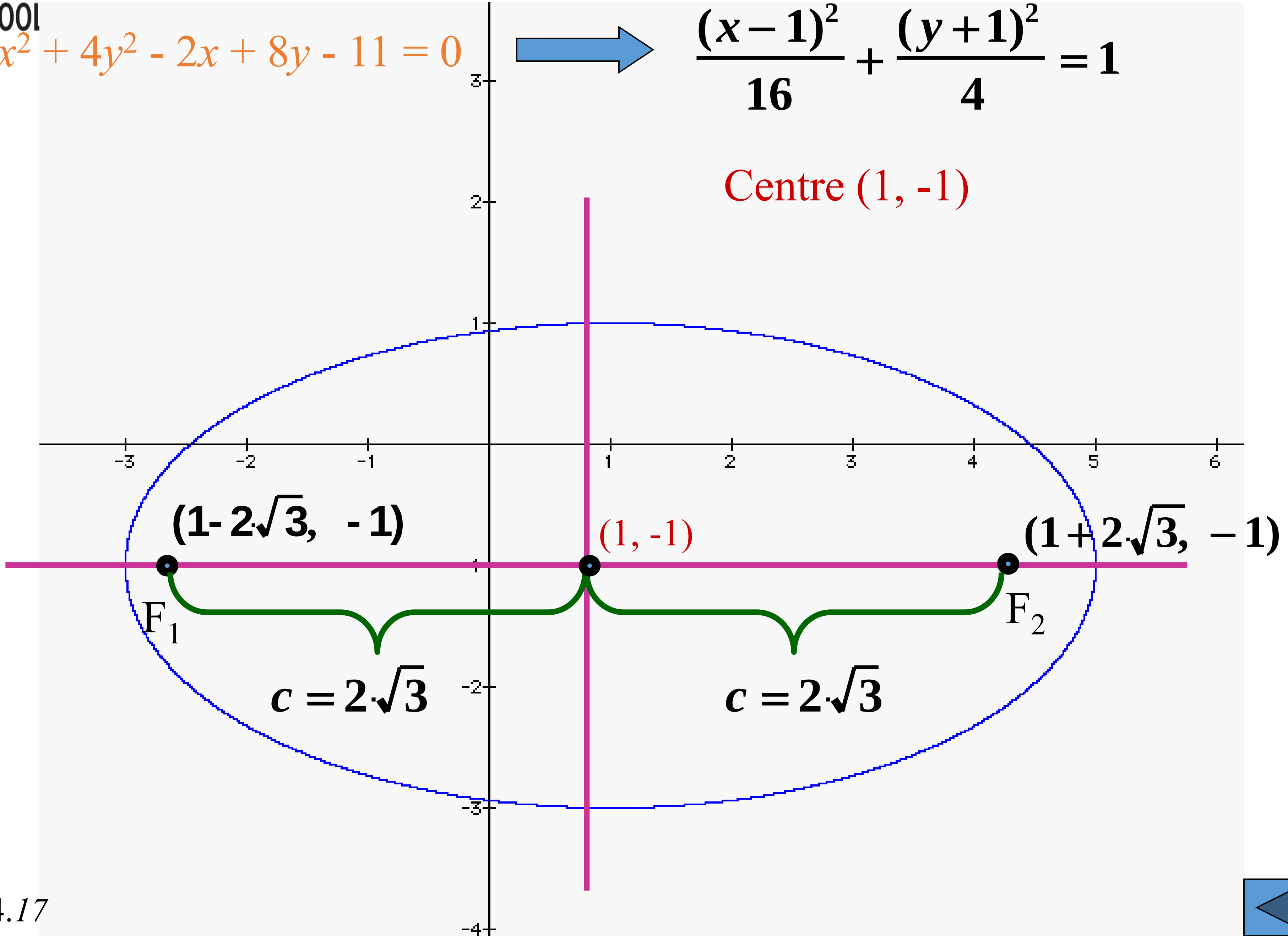
The centre is at (1, -1).
The major axis, parallel to the $x$-axis, has a length of 8 units.
The minor axis, parallel to the $y$-axis, has a length of 4 units.
The foci are at

$$(1 + 2\sqrt{3}, -1) \quad \text{and} \quad (1 - 2\sqrt{3}, -1).$$

# Sketching the Graph of the Ellipse [cont'd]

$x^2 + 4y^2 - 2x + 8y - 11 = 0$ → $\dfrac{(x-1)^2}{16} + \dfrac{(y+1)^2}{4} = 1$

Centre (1, -1)



# Analysis of the Ellipse

b) $9x^2 + 4y^2 - 18x + 40y - 35 = 0$

$$9x^2 + 4y^2 - 18x + 40y - 35 = 0$$

$$(9x^2 - 18x) + (4y^2 + 40y) - 35 = 0$$

$$9(x^2 - 2x + \underline{\quad 1 \quad}) + 4(y^2 + 10y + \underline{\quad 25 \quad}) = 35 + \underline{\quad 9 \quad} + \underline{\quad 100 \quad}$$

$$9(x - 1)^2 + 4(y + 5)^2 = 144$$

Since the larger number occurs under the $y^2$, the major axis is parallel to the $y$-axis.

$$\frac{(x-1)^2}{16} + \frac{(y+5)^2}{36} = 1$$

$h = 1$
$k = -5$
$a = 6$
$b = 4$

$$c^2 = a^2 - b^2$$
$$= 6^2 - 4^2$$
$$= 36 - 16$$
$$= 20$$

$$c = \sqrt{20}$$

$$c = 2\sqrt{5}$$

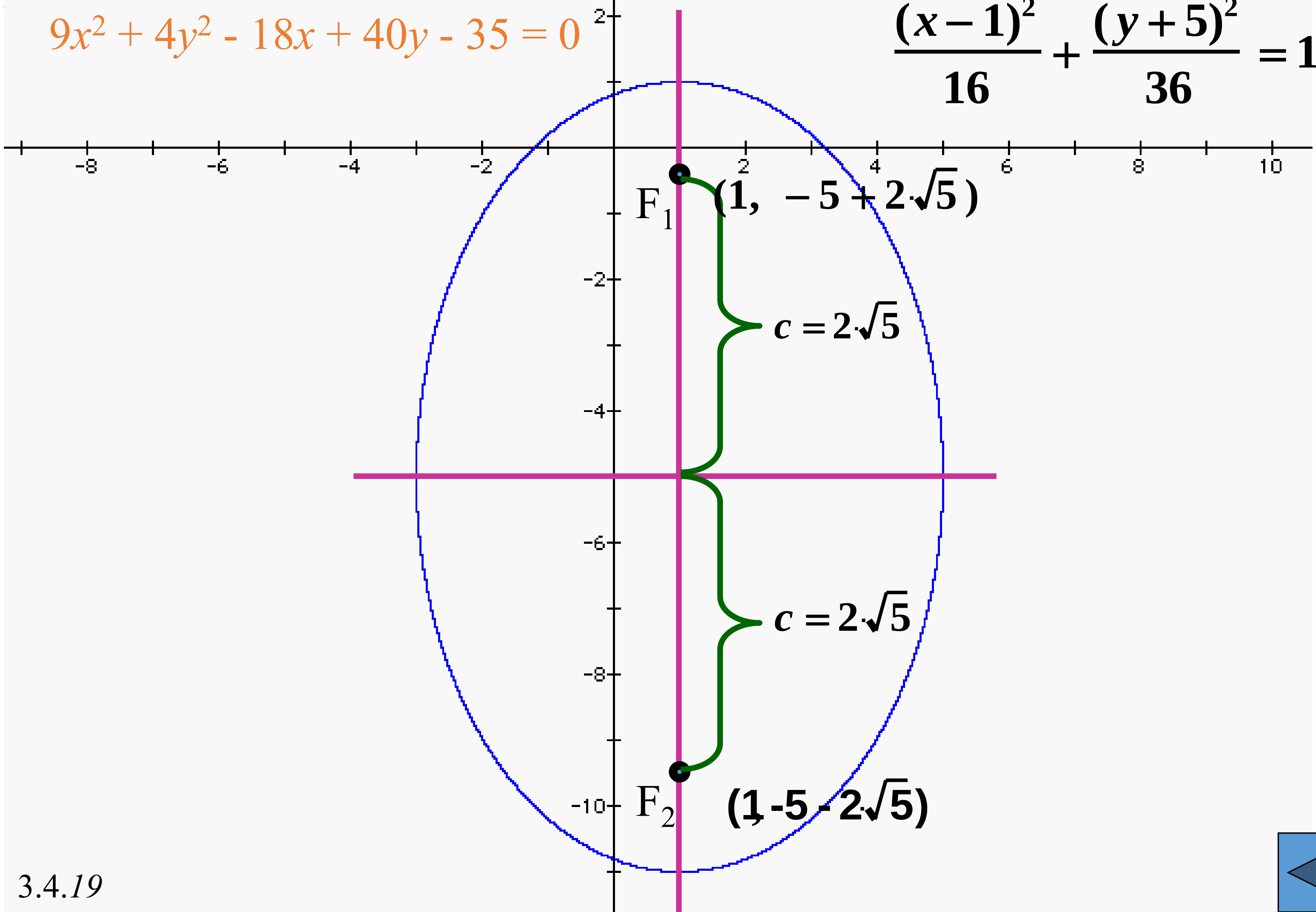
The centre is at (1, -5).
The major axis, parallel to the $y$-axis, has a length of 12 units.
The minor axis, parallel to the $x$-axis, has a length of 8 units.
The foci are at:

$$(1, -5 + 2\sqrt{5}) \quad \text{and} \quad (1, -5 - 2\sqrt{5})$$

# Sketching the Graph of the Ellipse [cont'd]

$9x^2 + 4y^2 - 18x + 40y - 35 = 0$

$$\frac{(x-1)^2}{16} + \frac{(y+5)^2}{36} = 1$$



# Graphing an Ellipse Using a Graphing Calculator

$$\frac{(x-1)^2}{16} + \frac{(y+1)^2}{4} = 1$$

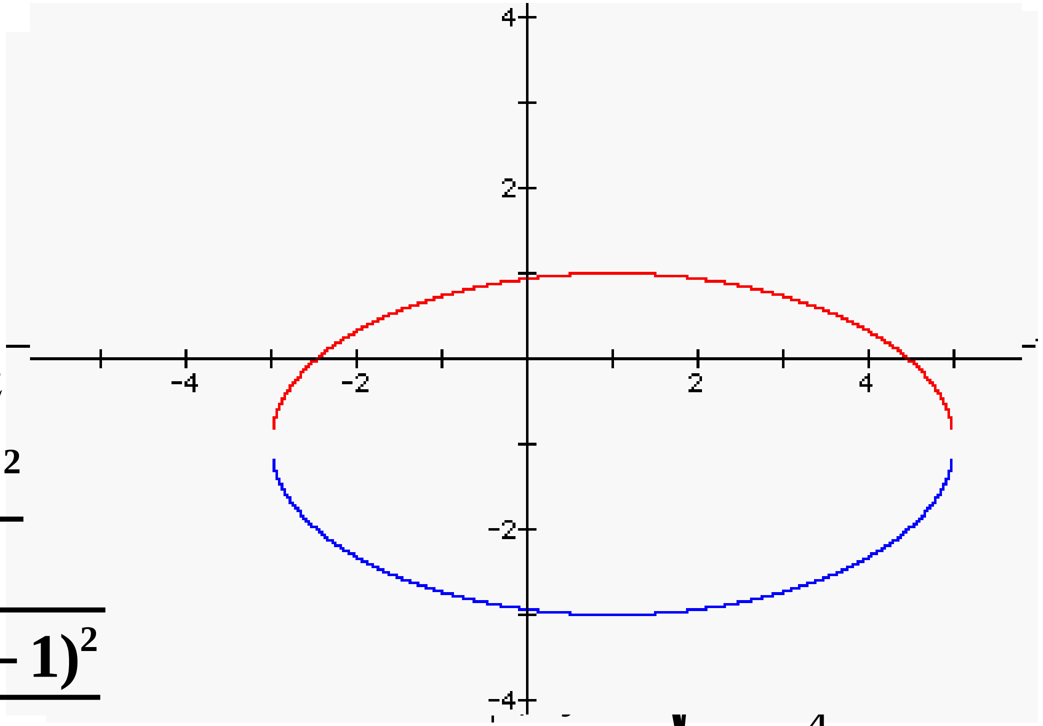
$$(x - 1)^2 + 4(y + 1)^2 = 16$$

$$4(y + 1)^2 = 16 - (x - 1)^2$$

$$(y+1)^2 = \frac{16-(x-1)^2}{4}$$

$$y+1 = \pm\sqrt{\frac{16-(x-1)^2}{4}}$$

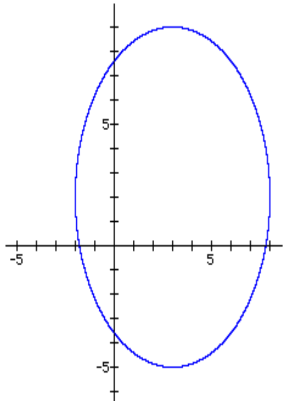
$$y = \pm\sqrt{\frac{16-(x-1)^2}{4}} - 1$$



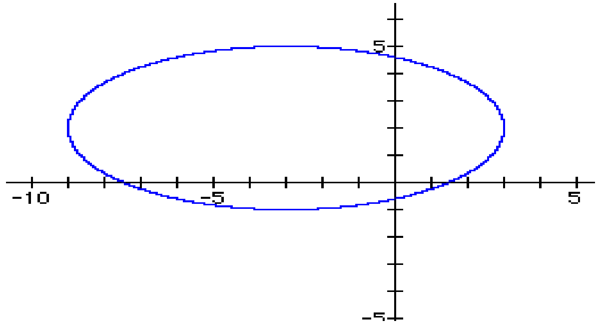
# General Effects of the Parameters $A$ and $C$

When $A \neq C$, and $A \times C > 0$, the resulting conic is an ellipse.

If $|A| > |C|$, it is a vertical ellipse.




If $|A| < |C|$, it is a horizontal ellipse.




The closer in value $A$ is to $C$, the closer the ellipse is to a circle.