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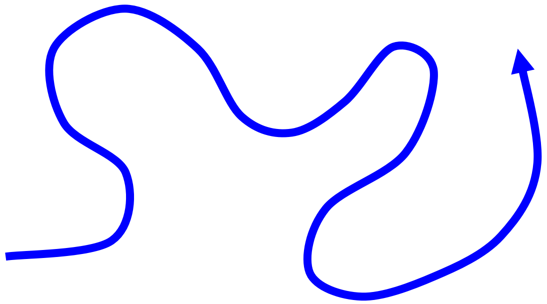
There are times when we need to describe motion (or a curve) that is *not a function*.



We can do this by writing equations for the x and y coordinates in terms of a third variable (usually $time$ or θ).

$$x = f(t) \quad y = g(t)$$

These are called <u>parametric equations</u>.

"t" is the *parameter*. (It is also the independent variable)

Picture this like a particle moving along and we know its x position over time and its y position over time.

We can figure out each of these and plot them together to see the movement of the particle.

Here are some examples of parametric functions

Example: A point is moving along the path parameterized by: $x(t) = 4 - t^2$; $y(t) = t^3 - 1$. What is the location of the point at time $t = 2$?

Given the parametric function
$$p(t) = (2at^2 + 6, at + 2bt - t)$$
and $p(2) = (-10, 10)$,
what are the values of a and b?

Graphing Parametric Equations

Parametric functions represent plane curves in which x and y are functions of the third variable t.

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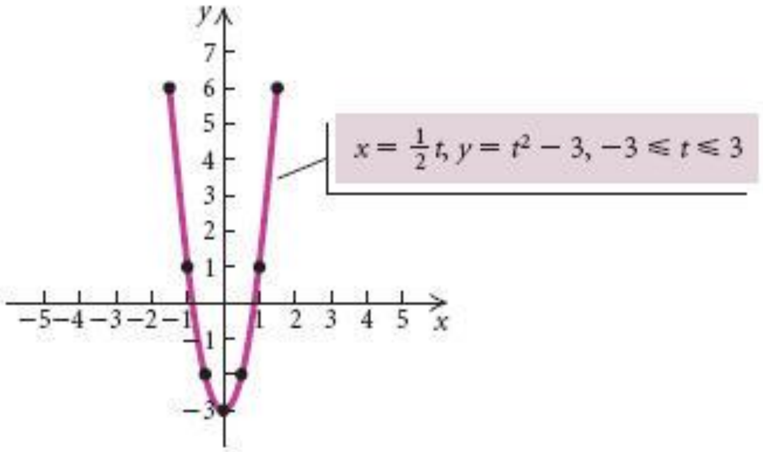
Example 1 Graphing a Parametric

Graph the curve represented by the equations

$$x = \frac{1}{2}t, \qquad y = t^2 - 3; \qquad -3 \le t \le 3.$$

t	x	y	(x, y)
-3	$-\frac{3}{2}$	6	$\left(-\frac{3}{2}, 6\right)$
-2	-1	1	$(-1, 1)$
-1	$-\frac{1}{2}$	-2	$\left(-\frac{1}{2}, -2\right)$
0	0	-3	$(0, -3)$
1	$\frac{1}{2}$	-2	$\left(\frac{1}{2}, -2\right)$
2	1	1	$(1, 1)$
3	$\frac{3}{2}$	6	$\left(\frac{3}{2}, 6\right)$



The rectangular equation is $y = 4x^2 - 3$

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Determining a Rectangular Equation for Given Parametric Equations
"Eliminating the Parameter"

1. Solve either equation for t.
2. Then substitute that value of t into the other equation.
3. Calculate the restrictions on the variables x and y based on the restrictions on t.

$$x = \frac{1}{2}t, \qquad y = t^2 - 3 \quad \longrightarrow \quad y = 4x^2 - 3$$

Find Rectangular Equation

Find a rectangular equation equivalent to

$$x = t^2, \quad y = t - 1$$

$y = t - 1$

$t = y + 1$

Substitute $t = y + 1$ into $x = t^2$.

$x = (y + 1)^2$

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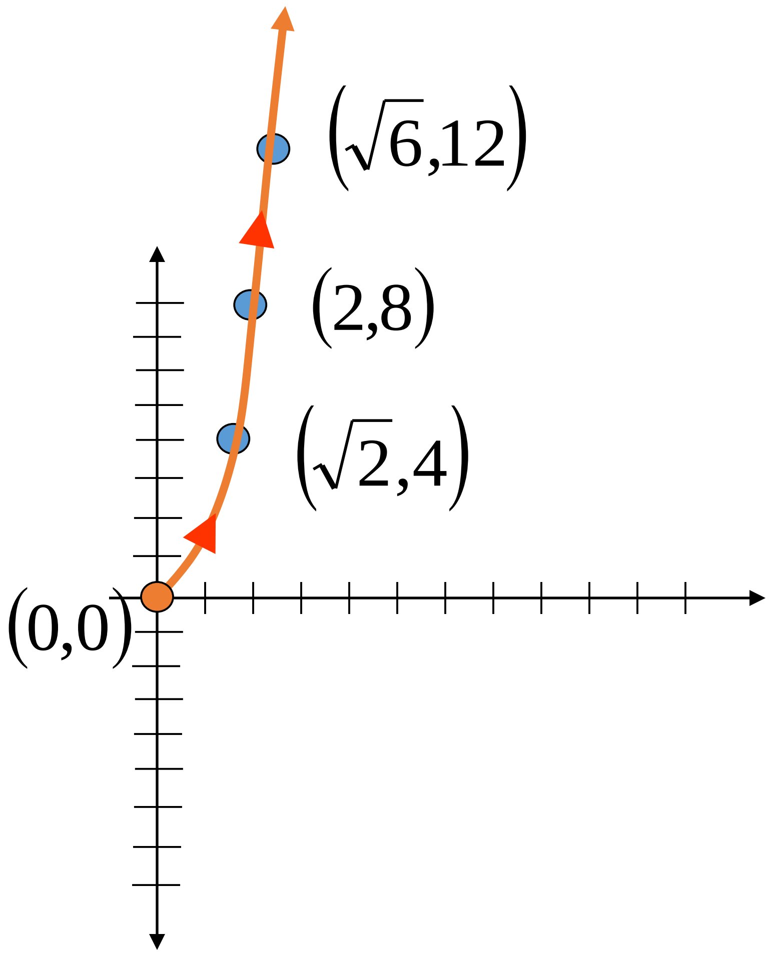
Graph the plane curve represented by the parametric equations

$$x = \sqrt{2t}, \quad y = 4t; \quad t \geq 0$$

t	x	y	(x, y)
0	$\sqrt{2(0)} = 0$	$4(0) = 0$	$(0,0)$
1	$\sqrt{2(1)} \approx 1.4$	$4(1) = 4$	$(\sqrt{2},4)$
2	$\sqrt{2(2)} = 2$	$4(2) = 8$	$(2,8)$
3	$\sqrt{2(3)} \approx 2.4$	$4(3) = 12$	$(\sqrt{6},12)$



Now, eliminate the parameter $x = \sqrt{2t}, \quad y = 4t; \quad t \ge 0$

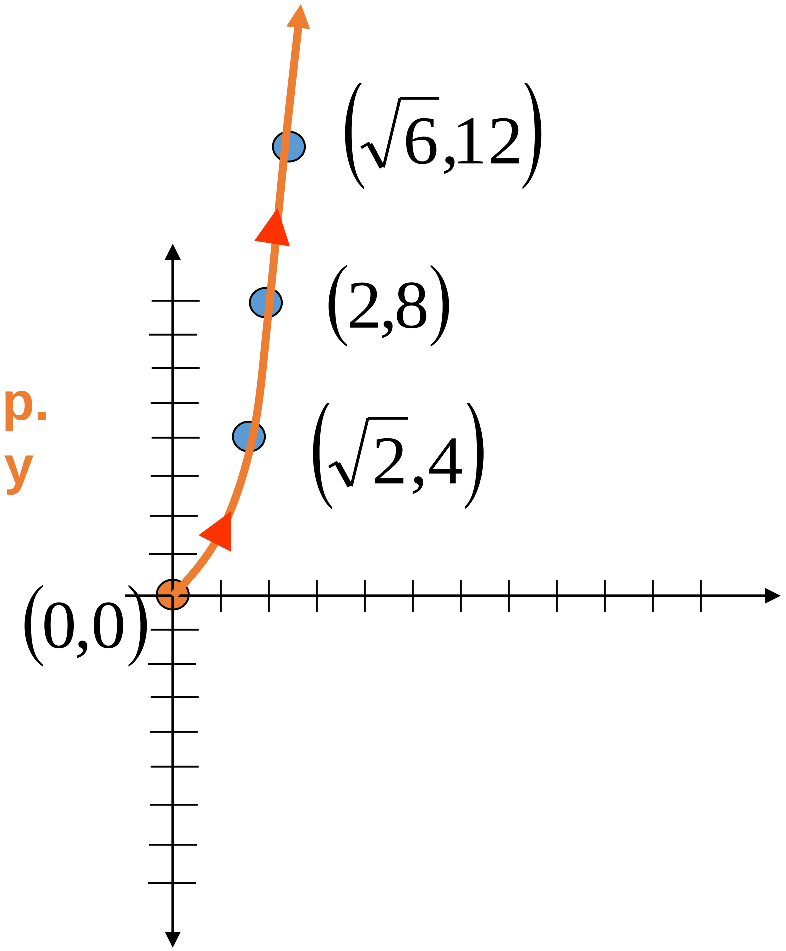
Solve for the parameter *t* in one of equations (whichever one is easier).

Substitute for *t* in the other equation.

$$x^2 = \frac{y}{2}$$

$$2x^2 = y$$

We recognize this as a parabola opening up. Since our domain for *t* started at 0, it is only the right half.



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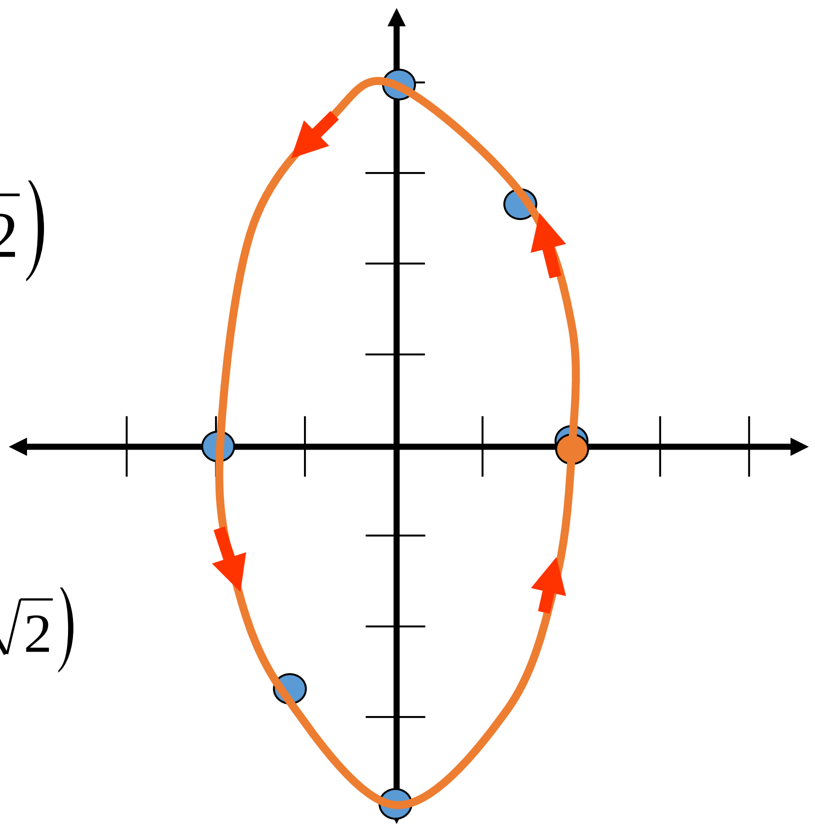
Graph the plane curve represented by the parametric equations

$$x = 2\cos t, \quad y = 4\sin t; \quad 0 \le t \le 2\pi$$

t	x	y	(x, y)
0	$2\cos 0 = 2$	$4\sin 0 = 0$	$(2,0)$
$\frac{\pi}{4}$	$2\cos\frac{\pi}{4} = \sqrt{2}$	$4\sin\frac{\pi}{4} = 2\sqrt{2}$	$(\sqrt{2}, 2\sqrt{2})$
$\frac{\pi}{2}$	$2\cos\frac{\pi}{2} = 0$	$4\sin\frac{\pi}{2} = 4$	$(0,4)$
π	$2\cos\pi = -2$	$4\sin\pi = 0$	$(-2,0)$
$\frac{5\pi}{4}$	$2\cos\frac{5\pi}{4} = -\sqrt{2}$	$4\sin\frac{5\pi}{4} = -2\sqrt{2}$	$(-\sqrt{2}, -2\sqrt{2})$
$\frac{3\pi}{2}$	$2\cos\frac{3\pi}{2} = 0$	$4\sin\frac{3\pi}{2} = -4$	$(0,-4)$



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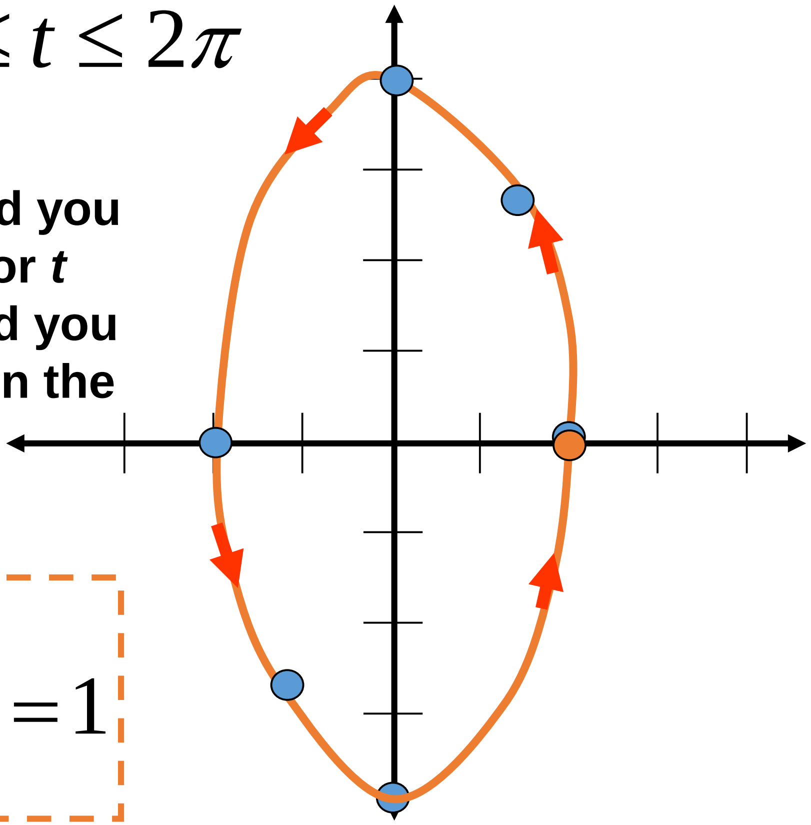
Now, eliminate the parameter. Based on our curve we'd expect to get the equation of an ellipse.

$$\frac{x}{2} = \frac{2\cos t}{2}, \quad \frac{y}{4} = \frac{4\sin t}{4}; \quad 0 \le t \le 2\pi$$



When you want to eliminate the parameter and you have trig functions, it is not easy to solve for *t* because it requires inverse functions. Instead you solve for cos *t* and sin *t* and substitute them in the Pythagorean Identity:

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{y}{4}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$$

$$\frac{y^2}{16} + \frac{x^2}{4} = 1$$

$$\text{From above}: \frac{y}{4} = \sin t \quad \frac{x}{2} = \cos t$$

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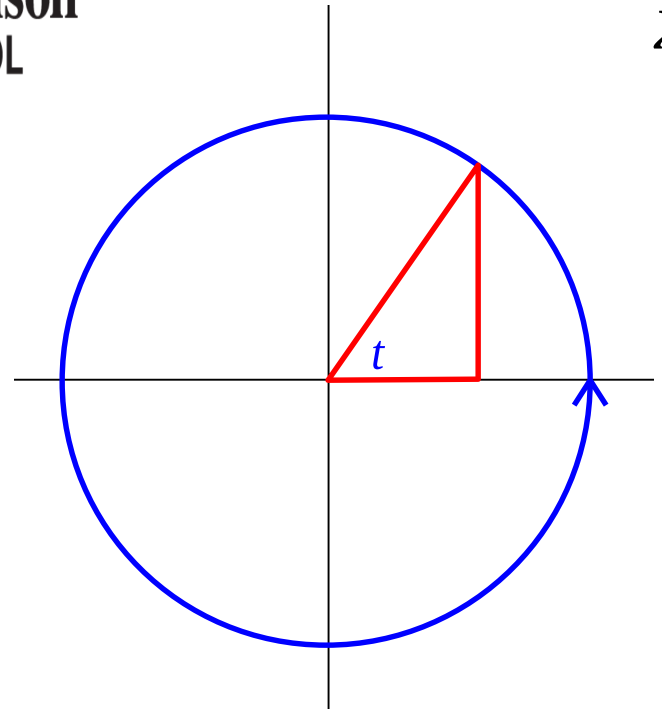
$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$



If we let t = the angle on the unit circle, then:

Since: $\sin^2 t + \cos^2 t = 1$

$$y^2 + x^2 = 1$$

We could identify the parametric equations as a circle.

$$x^2 + y^2 = 1$$

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You try! Write the rectangular equation for these parametric equations $x = 3\cos t \qquad y = 4\sin t$

$$\frac{x}{3} = \cos t \qquad \frac{y}{4} = \sin t$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2 t + \sin^2 t$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

This is the equation of an ellipse.