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The Binomial Theorem

- Expand a power of a binomial using Pascal's triangle or factorial notation.
- Find a specific term of a binomial expansion.

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The binomial theorem is used to raise a binomial (a + b) to relatively large powers. To better understand the theorem consider the following powers of (a+b):

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

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Note the following patterns for the expansion of $(a+b)^n$

- 1. There are *n+1* terms, the first a^n and last b^n
- 2. The exponents of *a* decrease and exponents of *b* increase
- 3. The sum of the exponents of *a* and *b* in each term is *n*

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Using these patterns the expansion of

$$(a+b)^8$$

looks like

$$(a+b)^8 =$$

$$a^8 + ?a^7b + ?a^6b^2 + ?a^5b^3 + ?a^4b^4 +$$

and the problem now comes down to finding the value of each coefficient.

This can be done using Pascal's triangle.

$(a+b)^0$ 1

$(a+b)^1$ 1 1

$(a+b)^2$ 1 2 1

$(a+b)^3$ 1 3 3 1

$(a+b)^4$ 1 4 6 4 1

$(a+b)^5$ 1 5 10 10 5 1

Pascal’s Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

The Binomial Theorem Using Pascal's Triangle

The Binomial Theorem Using Pascal's Triangle

For any binomial $a + b$ and any natural number n,

$$(a+b)^n = c_0a^nb^0 + c_1a^{n-1}b^1 + c_2a^{n-2}b^2 + \cdots + c_{n-1}a^1b^{n-1} + c_na^0b^n,$$

where the numbers $c_0, c_1, c_2, \ldots, c_{n-1}, c_n$ are from the $(n + 1)$st row of Pascal's triangle.

Example

- Expand $(u - v)^4$.

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Another Example

- Expand $(x-3y)^4$.
- $a = x$, $b = -3y$, and $n = 4$. We use the 5th row of Pascal's triangle: **1 4 6 4 1**
- Then we have

$$=1(x)^4+4(x)^3(-3y)^1+6(x)^2(-3y)^2+4(x)(-3y)^3+1(-3y)^4$$
$$=x^4-12x^3y+54x^2y^2-108xy^3+81y^4$$

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Although Pascal’s triangle can be used to expand a binomial, as the value of the exponent gets larger, it becomes more and more tedious to use this method. The binomial theorem is used for these larger expansions. Before proceeding to the theorem we need some additional notation.

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The product of the first n natural numbers is denoted $n!$ and is called n **factorial**.

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

and $0! = 1$

$$5! = (1)(2)(3)(4)(5) = 120$$

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The ***binomial coefficient***:

let n and r be nonnegative integers with $r \leq n$

The ***binomial coefficient*** is denoted by

and

$${}_nC_r \text{ or } \binom{n}{r}$$

is defined by

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

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Evaluate the expression:

$$\frac{5!}{2!3!}\binom{8}{3}\binom{6}{3}\binom{8}{5}$$

$$\text{In general}\binom{n}{r}=\binom{n}{n-r}$$

The Binomial Theorem Using Factorial Notation

The Binomial Theorem Using Factorial Notation

For any binomial $a + b$ and any natural number n,

Use binomial theorem to find

$$(3a-2b)^6$$

$$\overset{1}{\binom{6}{0}}(3a)^6+\overset{6}{\binom{6}{1}}(3a)^5(-2b)^1+\overset{15}{\binom{6}{2}}(3a)^4(-2b)^2+\overset{20}{\binom{6}{3}}(3a)^3(-2b)^3+$$

$$\overset{15}{\binom{6}{4}}(3a)^2(-2b)^4+\overset{6}{\binom{6}{5}}(3a)^1(-2b)^5+\overset{1}{\binom{6}{6}}(-2b)^6$$

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Example

EXAMPLE 4 Expand: $\left(\frac{2}{x} + 3\sqrt{x}\right)^4$.

Solution We have $(a + b)^n$, where $a = 2/x$, $b = 3\sqrt{x}$, and $n = 4$. Then using the binomial theorem, we have

Finding a Specific Term

- **Finding the (r + 1)st Term**

  The (r + 1)st term of $(a + b)^n$ is

$$\binom{n}{r} a^{n-r} b^{r}.$$

Example: Find the 7^{th} term in the expansion $(x^2 -$

x^2, b