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The Derivative

One of the roots of Calculus was the problem of finding the slope of a line that is tangent to the graph of a curve at a point on the graph.

First, let's do a quick review of the meaning of slope of a line.

The slope of a line is the ratio of the vertical change to the horizontal change between two points on the line.

$$\text{slope} = m = \frac{\text{vertical change}}{\text{horizontal change}}$$

You may remember this as:

$$\text{slope} = m = \frac{\text{rise}}{\text{run}}$$

but…

Never lose track of the fact that rise means vertical change and run means horizontal change.

Now, how have you already learned how to calculate the slope of a line?

You need the coordinates of two points.

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Let's say that the graph of a line passes through the points (3, - 4) and (9, 5). What is the slope of the line?

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(5-(-4))}{(9-3)} = \frac{9}{6} = \frac{3}{2}$$

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From here, if you wanted to find the equation of the line that passes through those two points, you could use the point-slope form of a linear equation.

OK, now… back to tangent lines.

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You probably recall from Geometry class that a tangent line is a line that touches a graph at only one point.

This definition has a condition ("only one point") that we do not use in Calculus, but it will serve as a good starting point.

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Let's say that on the graph of a function, a line is tangent to the graph at the point (5, -2). How would you find the slope of the tangent line?

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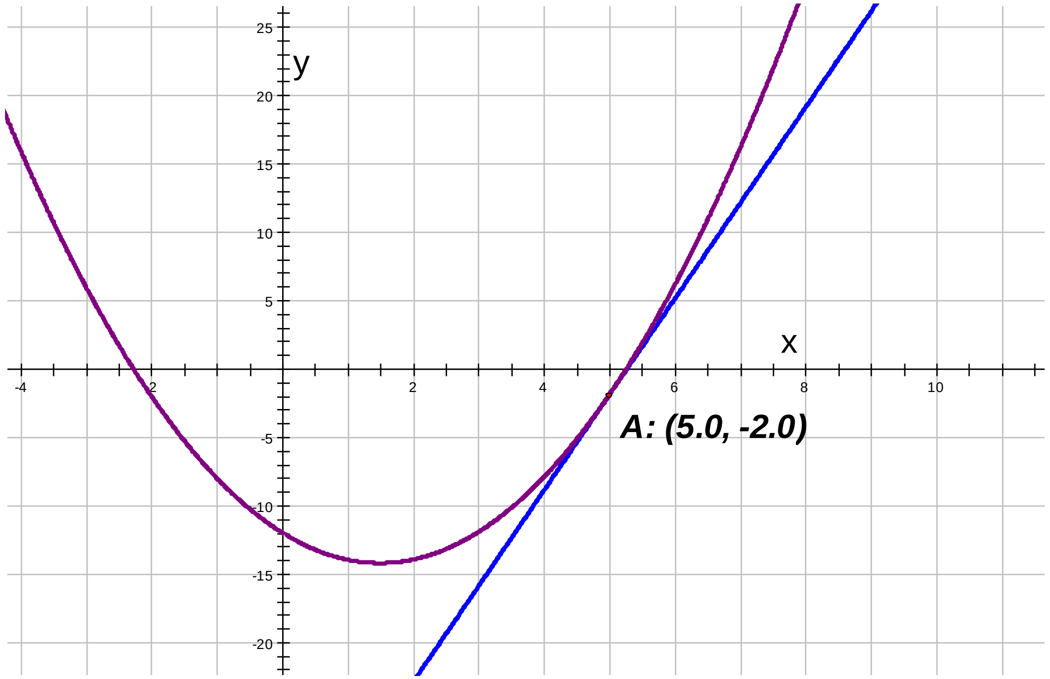
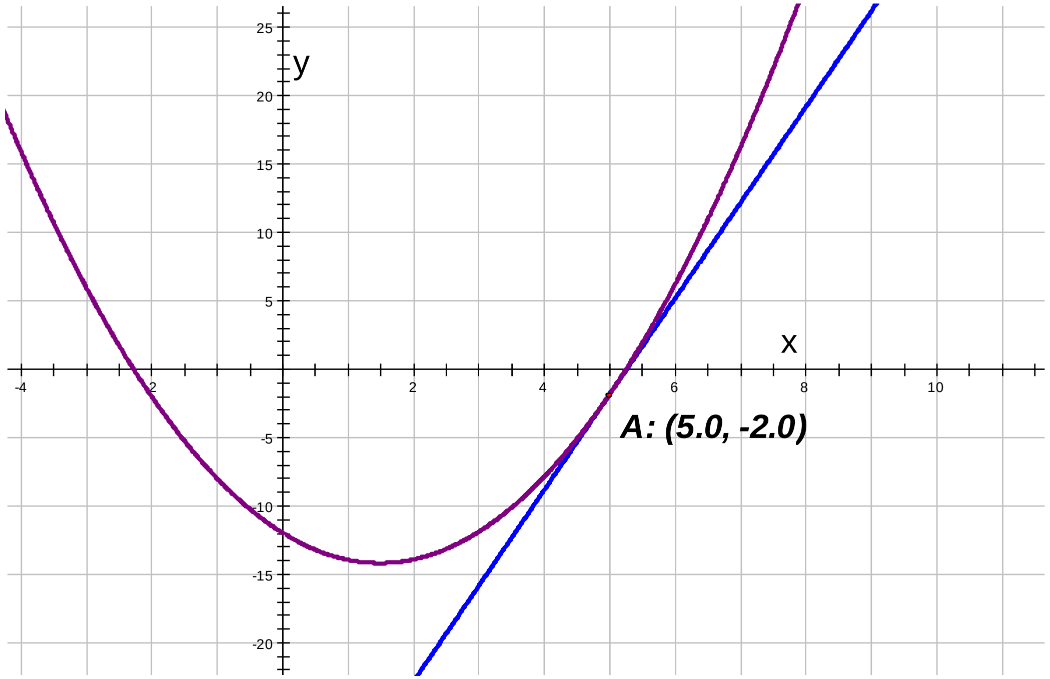
We have a problem in that we only know one point on the tangent line.



We need two points to calculate the slope using the definition of slope of a line that we currently have.

But this is Precalculus and we know about limits!



Consider a new approach to tangent lines.

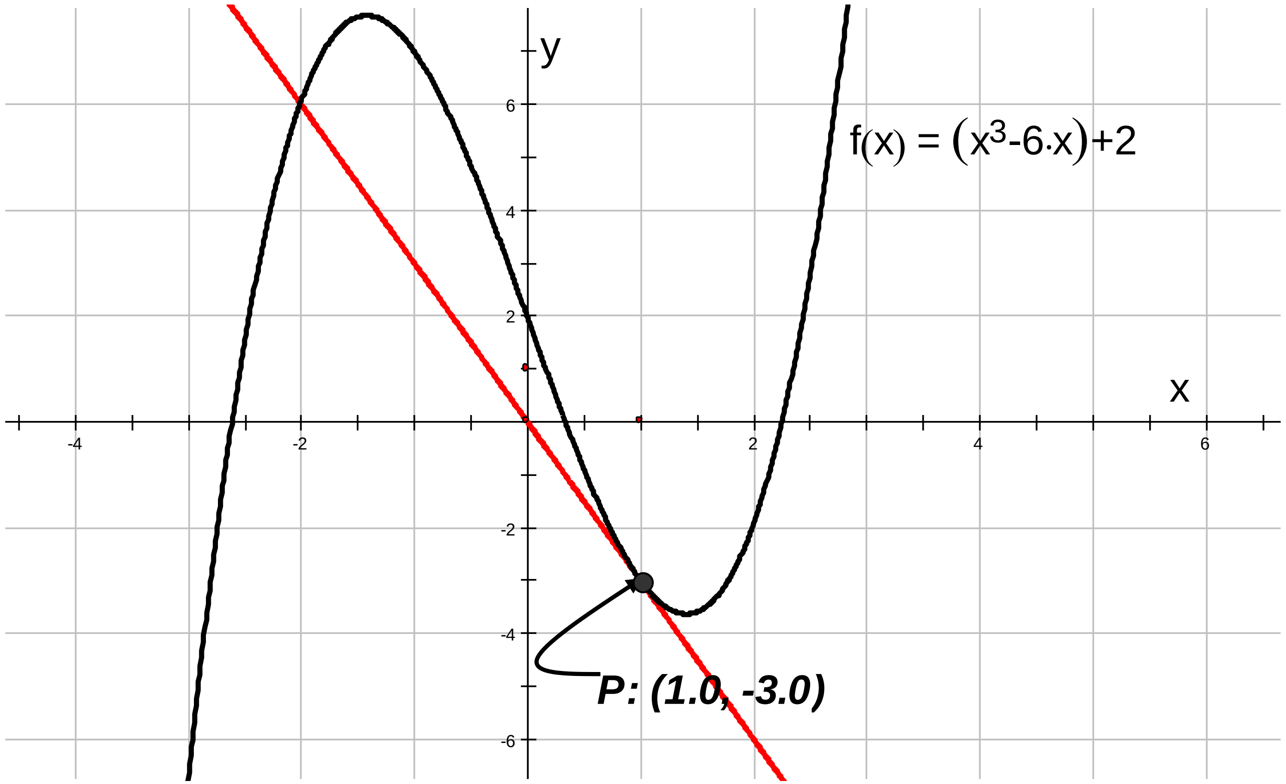
Let's use the function $f(x) = x^3 - 6x + 2$

for our exploration.

We will find the slope of the line that is tangent to the graph of the function at

(1, -3).

Let's take a look at the graphs of the function and the tangent line first.



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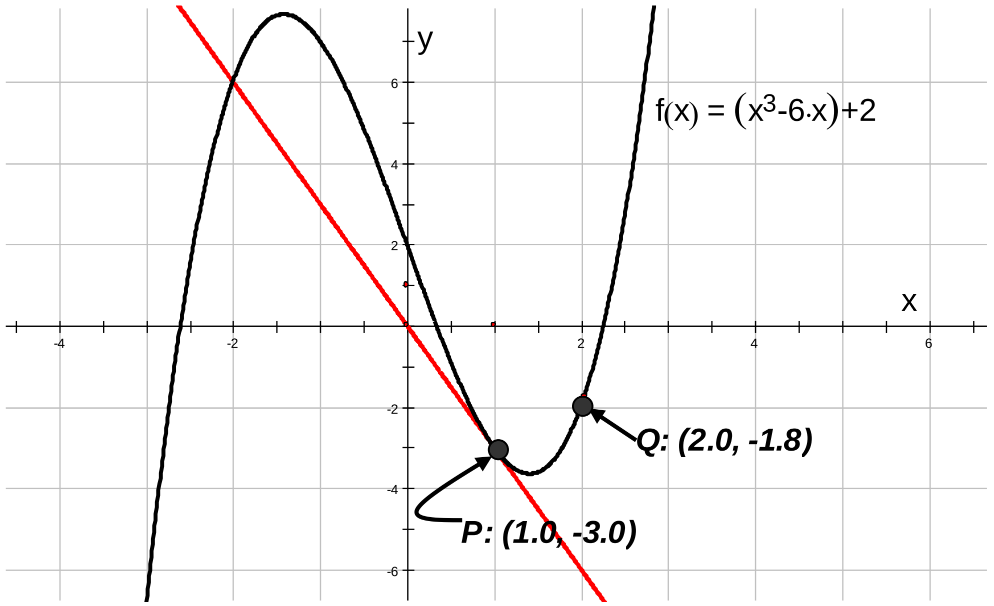
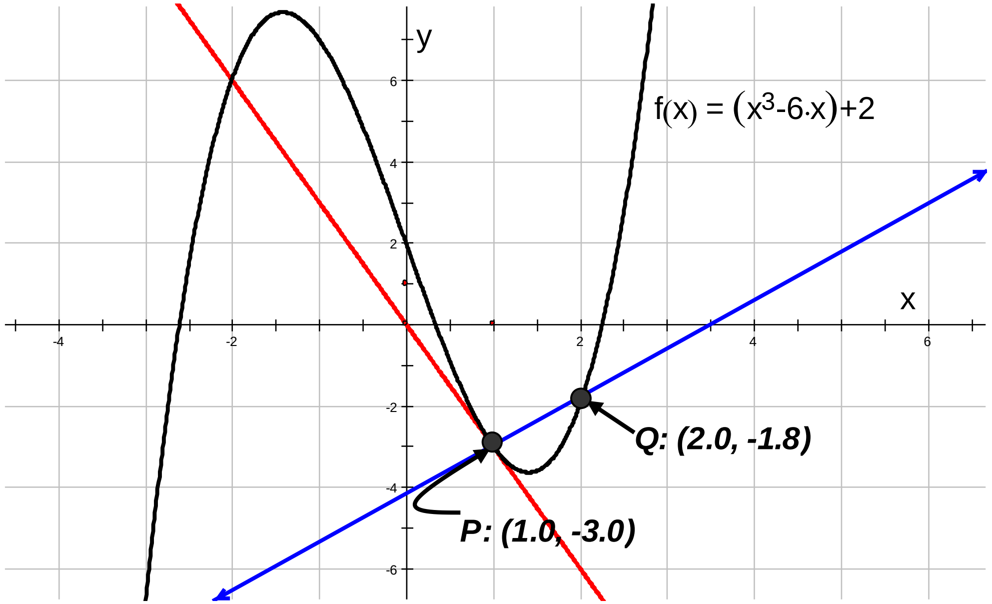
To define the tangent line at any point, P, on the graph of *f(x)*, let point Q be any other point on the graph.



The line that passes through the two points P and Q is called a secant line.

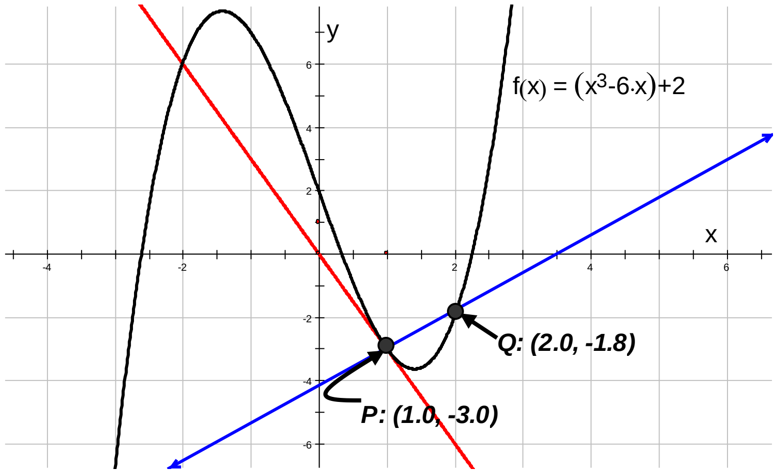


Because we know the coordinates of two points on the secant line, we can calculate its slope using the technique that we know from Geometry:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-1.8 - (-3)}{2 - 1}$$

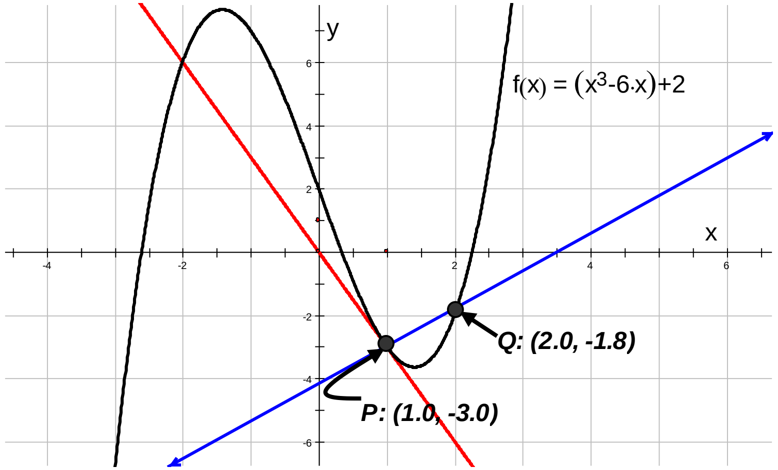
$$m = \frac{1.2}{1} = 1.2$$



Now that we have the slope of the secant line, can we us it as an approximation of what we really want to know - the slope of the tangent line?

Yes…

But will it be a good approximation?



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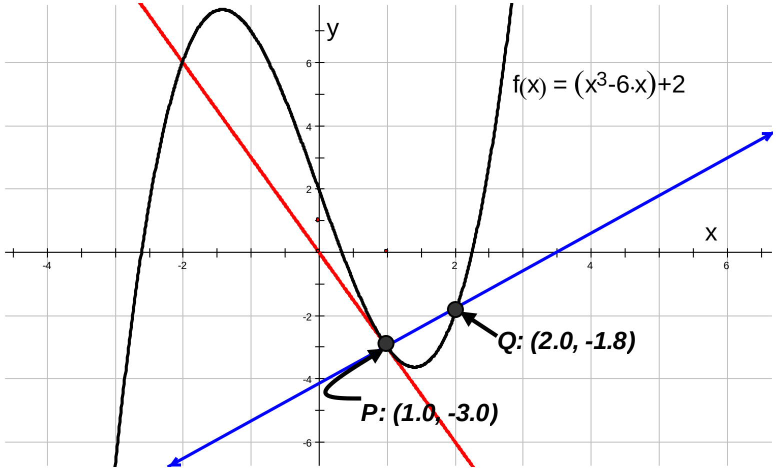
In this example, no.

The tangent line clearly has a negative slope and the slope of the secant line currently has a positive slope.

But what if we moved point Q along the curve closer to tangent point P?



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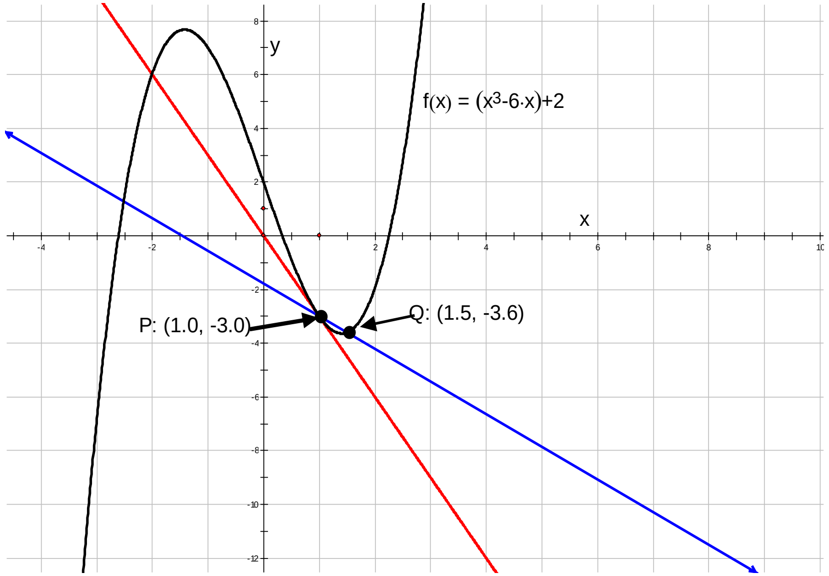
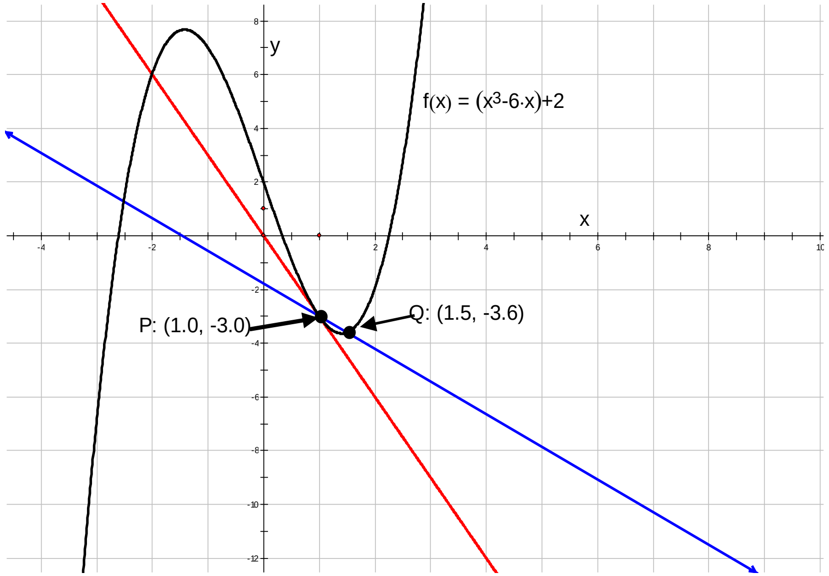
The secant line has changed position and it has the same orientation ("falling line") as the tangent line.

It's slope should be a better approximation of the slope of the tangent line.



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3.6-(-3)}{1.5-1} = \frac{-0.6}{0.5} = -1.2$$

Let's move point Q along the curve even closer to tangent point P.



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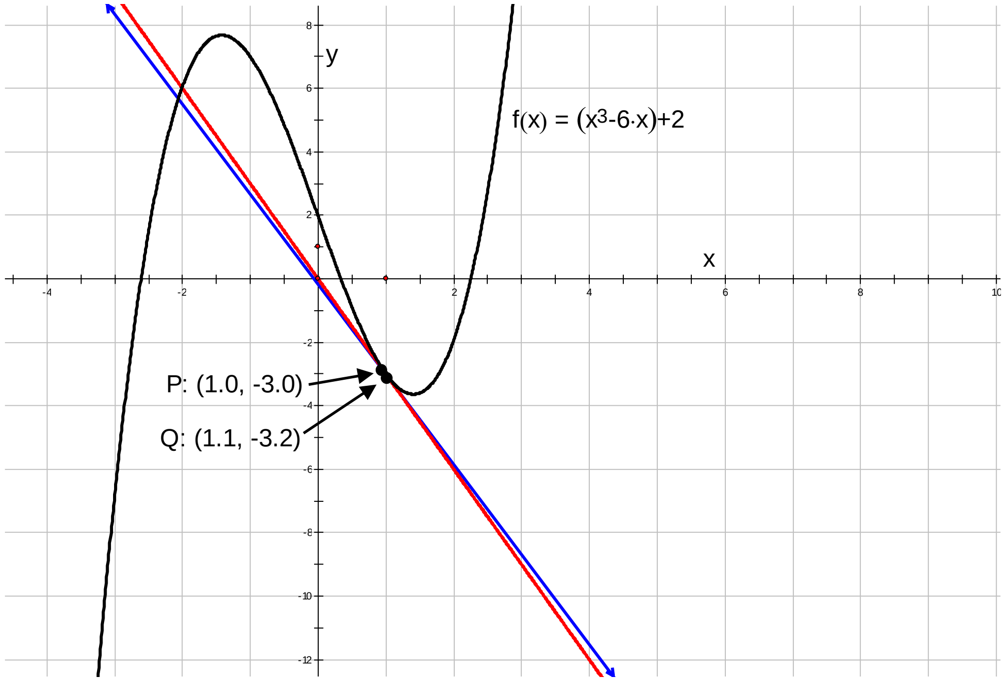
Now point Q is quite close to point P. As a result, the secant line is an even better approximation of the tangent line.



Let’s look at the slope of the secant line again.

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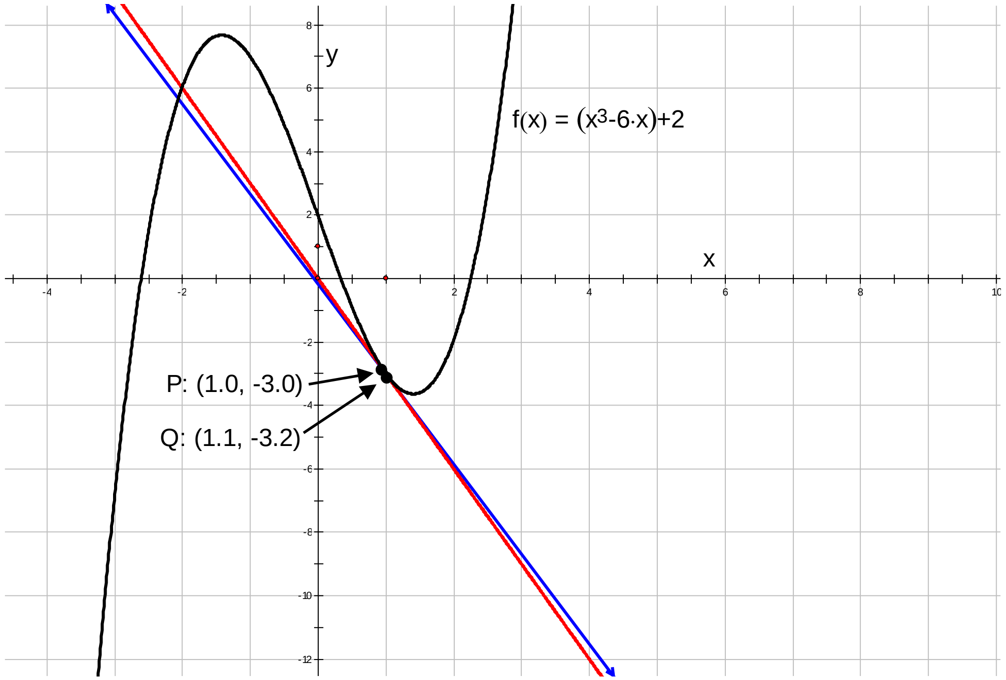
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-3.2 - (-3)}{1.1 - 1}$$

$$m = \frac{-0.2}{0.1} = -2$$



Let’s analyze what we have done in the last several slides.

We wished to find the slope of the tangent line to the graph of a function at a point.

Because we only knew one point on the tangent line, we could not use the slope formula from Geometry.

But we could use the slope formula with a secant line that passed through the tangent point and another point on the curve near the tangent point.

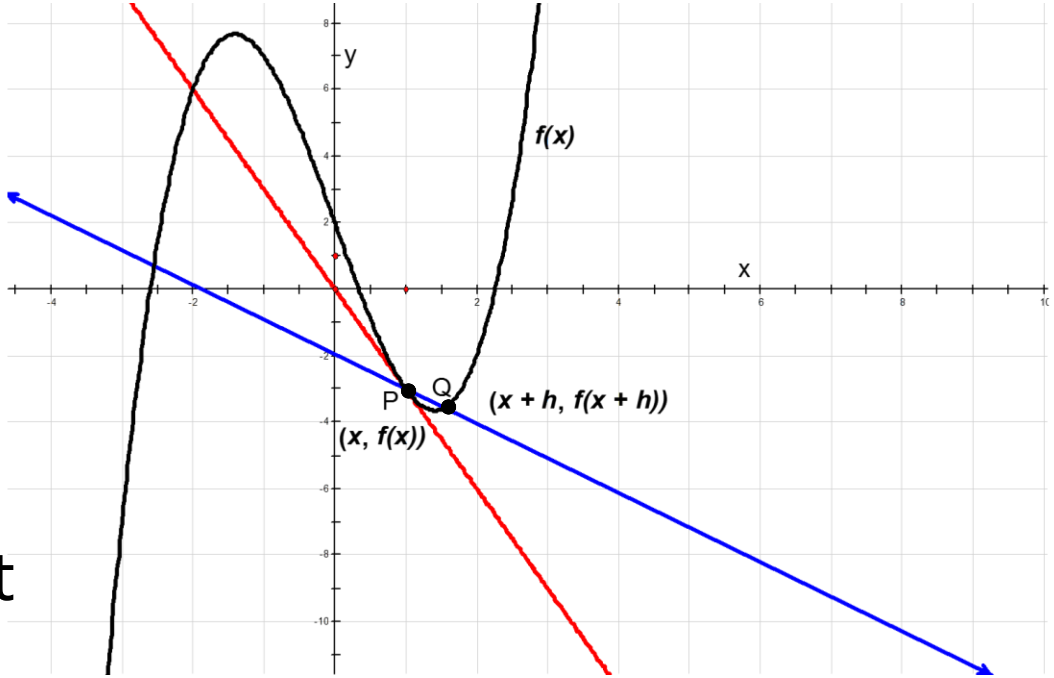
As we moved this second point closer to the tangent point, the position of the secant line became a better approximation of the tangent line and thus, the slope of the secant line became a better approximation of the slope of the tangent line.

We were using a limit process!

Let's label our function simply as f(x).

The coordinates of the tangent point P are **(x, f(x)).**

Since point Q is a point on the curve other than point P, it has a different set of coordinates.



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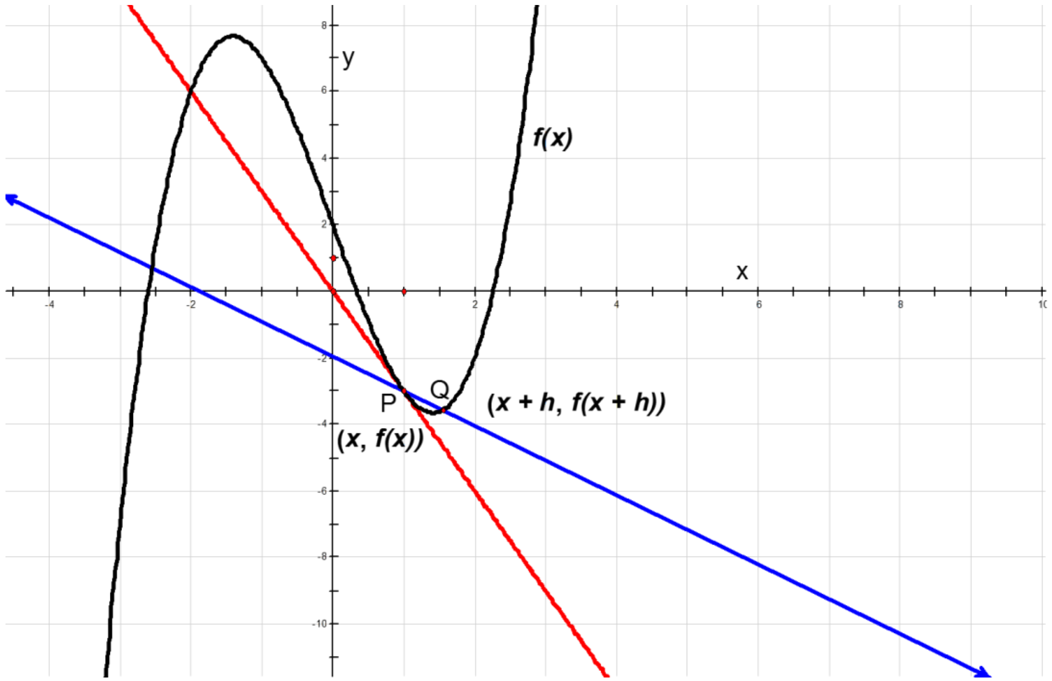
The x-coordinates of points P and Q differ by some amount. Let's label this difference as h.

The coordinates of point Q, then are **(x + h, f(x + h)).**



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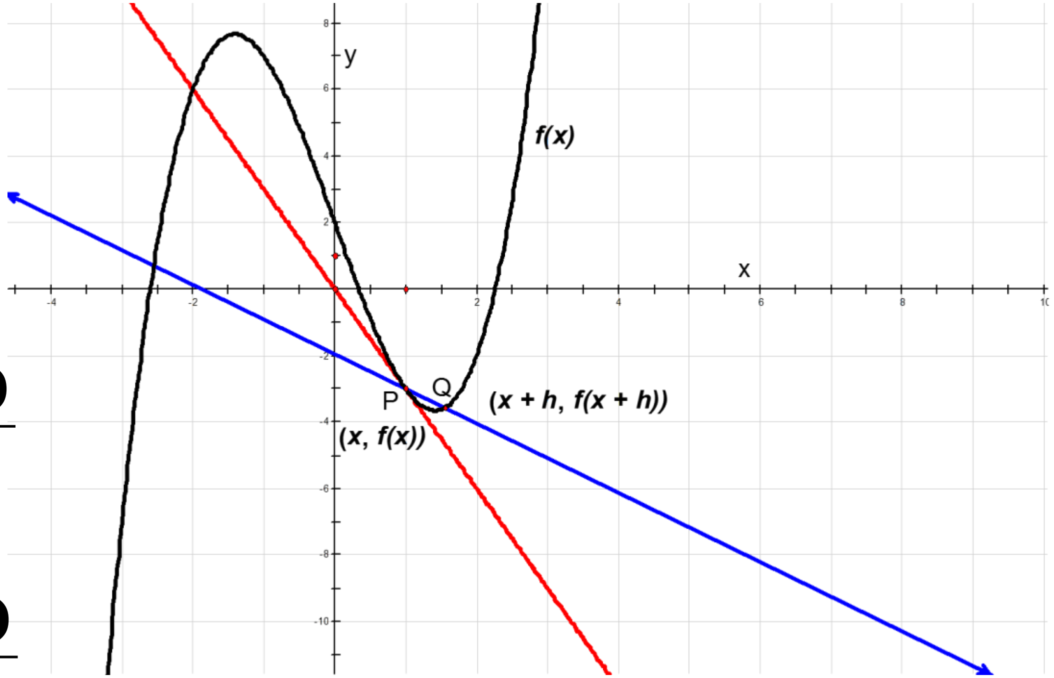
The slope of the secant line can be represented as follows:

$$m_{sec} = \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$m_{sec} = \frac{f(x+h) - f(x)}{h}$$



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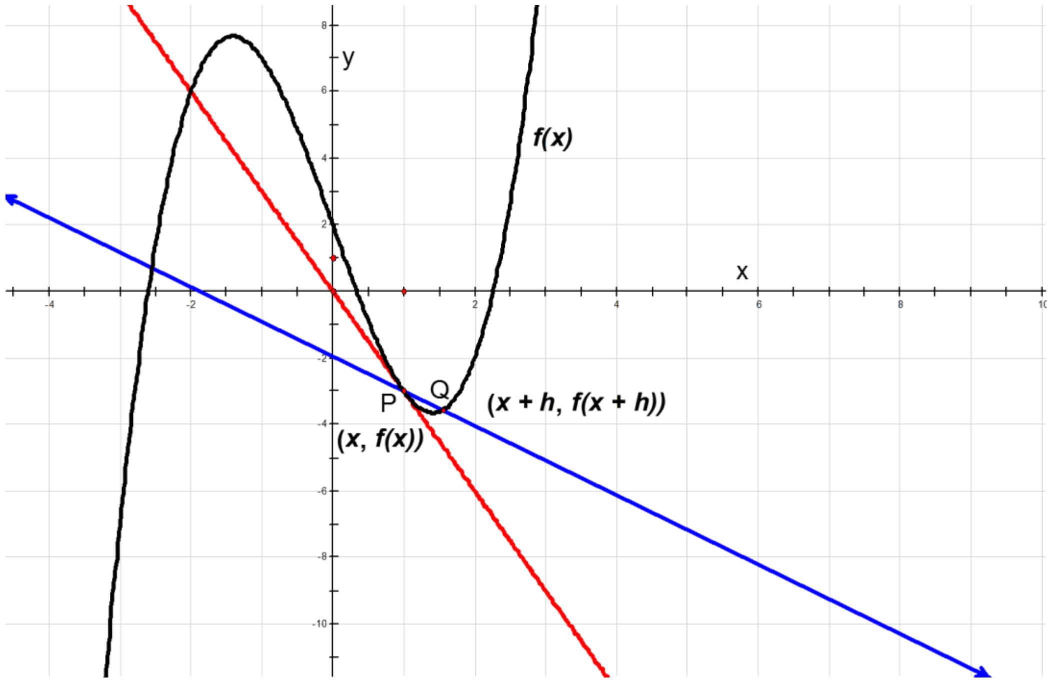
The closer point Q approaches the tangent point P, the better the slope of the secant line approximates the slope of the tangent line.



As point Q approaches point P, what happened to the value of h?

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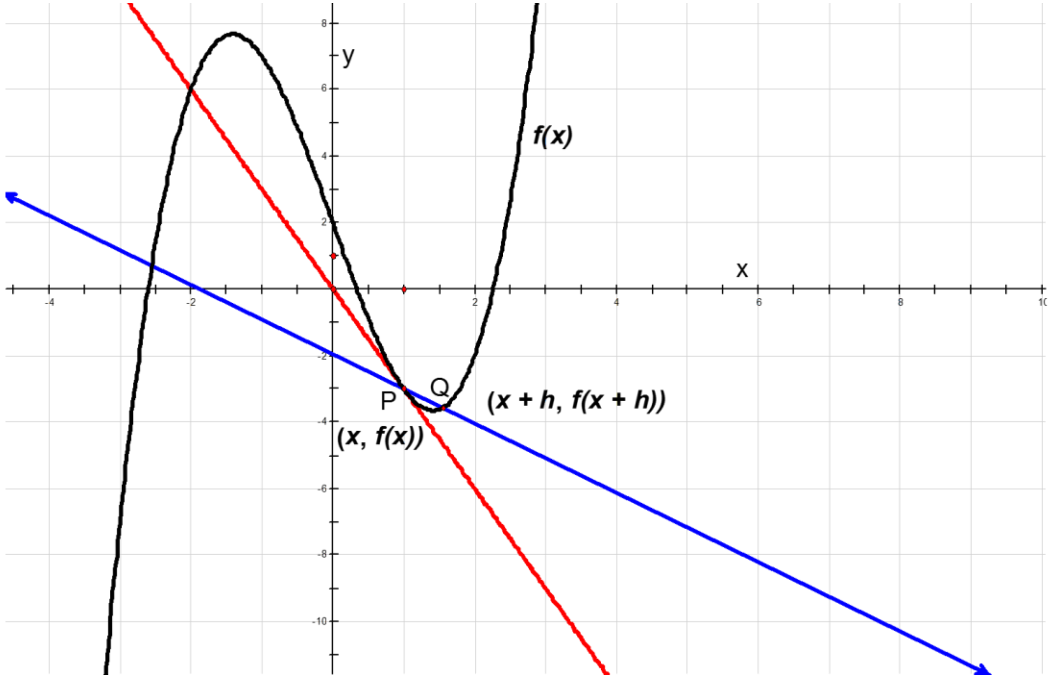
The value of h gets closer to zero.



We now *define the slope of the tangent line at point P (x, f(x)) as follows:*

$$m_{\tan} = \lim_{h \to 0} (m_{\sec}) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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The slope of the tangent line is the limit of the slope of the secant line as the value of h approaches zero.

The expression $\frac{f(x+h)-f(x)}{h}$ is so important in Precalculus and Calculus that it has a name.

It is called the difference quotient.

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The difference quotient measures the average rate of change between two points. It is the slope of a secant line to a graph.

The limit of the difference quotient as h approaches zero measures the instantaneous rate of change at the point of tangency. It is the slope of the tangent line to a graph at the point of tangency.

It is the slope of the tangent line to a graph at the point of tangency.

The instantaneous rate of change of a function (the slope of the tangent line) also has a very special name in Precalculus and Calculus.

It is called the derivative of the function at x.

derivative of f(x)= $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

There are several ways to represent the derivative of function f(x). Two of the most common are:

$$f'(x) \quad \text{and} \quad \frac{df}{dx}$$

Let's calculate the slope of the tangent line to the function that we started with earlier in the exploration:

$$f(x) = x^3 - 6x + 2$$

We need to find the instantaneous rate of change of f(x). We need to find the **derivative of f(x).**

We need to calculate $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

We can do this in four steps:

1. Compute f(x + h).
2. Form the difference f(x + h) – f(x).
3. Form the quotient $\frac{f(x+h)-f(x)}{h}$.
4. Compute f′(x) = $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

When you get comfortable with the process and the Algebra required in it, you can combine the first three steps into one step. For now, we'll take small steps.

$$f(x) = x^3 - 6x + 2$$

1. $$f(x+h) = (x+h)^3 - 6(x+h) + 2$$
$$= x^3 + 3x^2h + 3xh^2 + h^3 - 6x - 6h + 2$$

2. $$f(x+h) - f(x) = x^3 + 3x^2h + 3xh^2 + h^3 - 6x - 6h + 2 - \left(x^3 - 6x + 2\right)$$
$$= x^3 + 3x^2h + 3xh^2 + h^3 - 6x - 6h + 2 - x^3 + 6x - 2$$
$$= 3x^2h + 3xh^2 + h^3 - 6h$$

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3. $$\frac{f(x+h)-f(x)}{h} = \frac{3x^2h+3xh^2+h^3-6h}{h}$$

4. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-6h}{h}$$

$$= \lim_{h\to 0}\frac{h\left(3x^2+3xh+h^2-6\right)}{h} = \lim_{h\to 0}\left(3x^2+3xh+h^2-6\right)$$

$$= 3x^2-6$$

Remember, the derivative of a function is another function.

In the example that we just worked,

the derivative of the function $f(x) = x^3 - 6x + 2$

is the function $f'(x) = 3x^2 - 6$.

The derivative function allows you to calculate the instantaneous rate of change of a function at any point at which the function is defined.

All that you need to do is to evaluate the derivative function at the given point.

So, to complete our exploration and find the slope of the tangent line of the function $f(x) = x^3 - 6x + 2$ at the point (1, -3), we find the value of the derivative function, $f'(x) = 3x^2 - 6$, at x = 1.

$$f(x) = x^3 - 6x + 2$$

$$f'(x) = 3x^2 - 6$$

$$f'(1) = 3(1)^2 - 6 = 3 - 6 = -3$$

Therefore, the slope of the tangent line to the function $f(x) = x^3 - 6x + 2$ at the point (1, -3), is $f'(1) = -3$.