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Limits and Continuity

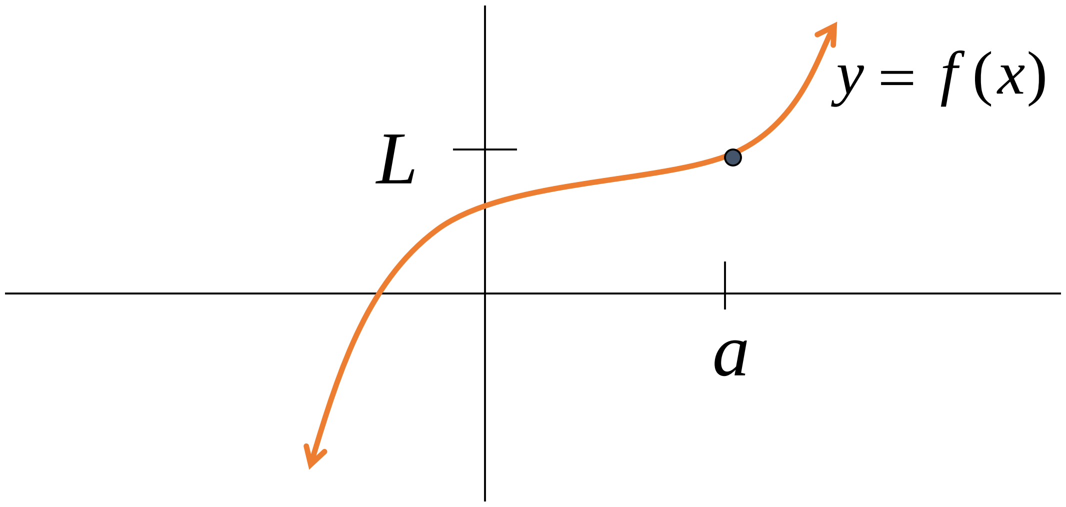
- Definition
- Evaluation of Limits
- Continuity
- Limits Involving Infinity

Limit

We say that the limit of $f(x)$ as x approaches a is L and write

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ approach L as x approaches a.



Limits, Graphs, and Calculators

1. a) Use table of values to guess the value of $\lim_{x \to 1} \left(\frac{x-1}{x^2-1} \right)$

   b) Use your calculator to draw the graph $f(x) = \frac{x-1}{x^2-1}$ and confirm your guess in a)

   Graph 1

2. Find the following limits

   a) $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)$ by considering the values

$x = \pm 1, \pm 0.5, \pm 0.1, \pm 0.05, \pm 0.001.$ Thus the limit is 1.

Confirm this by ploting the graph of $f(x) = \frac{\sin x}{x}$

Graph 2

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b) $\lim_{x \to 0} \sin\left(\pi/x\right)$ by considering the values

(i) $x = \pm 1,\ \pm 1/10,\ \pm 1/100,\ \pm 1/1000$

(ii) $x = \pm 1,\ \pm 2/3,\ \pm 2/103,\ \pm 2/1003$

This shows the limit does not exist.

Confirm this by ploting the graph of $f(x) = \sin\left(\pi/x\right)$

Graph 3

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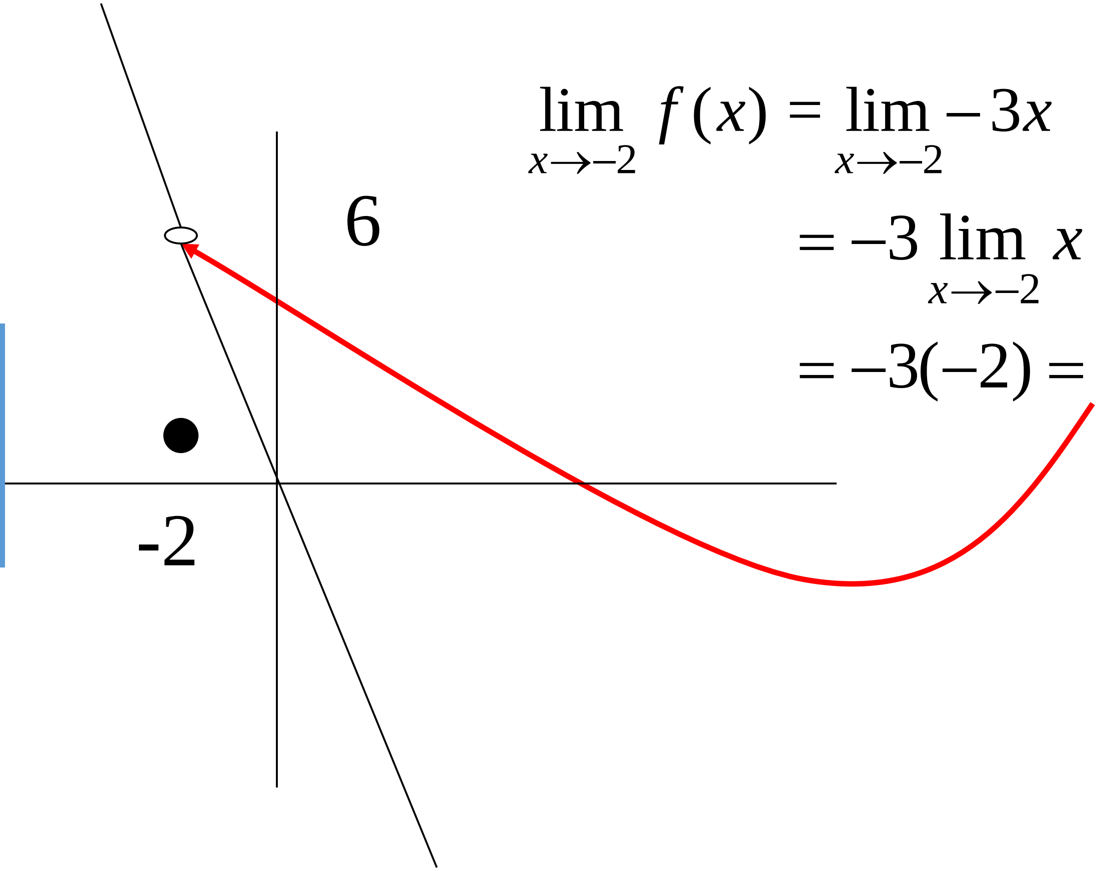
c) Find $\lim\limits_{x \to -2} f(x)$ where $f(x) = \begin{cases} -3x & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} -3x$$

$$= -3 \lim_{x \to -2} x$$

$$= -3(-2) = 6$$



Note: f(-2) = 1

is not involved

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3) Use your calculator to evaluate the limits

a. $\lim_{x \to 2} \left(\frac{4(x^2 - 4)}{x - 2} \right)$ Answer : 16

b. $\lim_{x \to 0} g(x)$, where $g(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$ Answer : no limit

c. $\lim_{x \to 0} f(x)$, where $\mathrm{f}(x) = \frac{1}{x^2}$ Answer : no limit

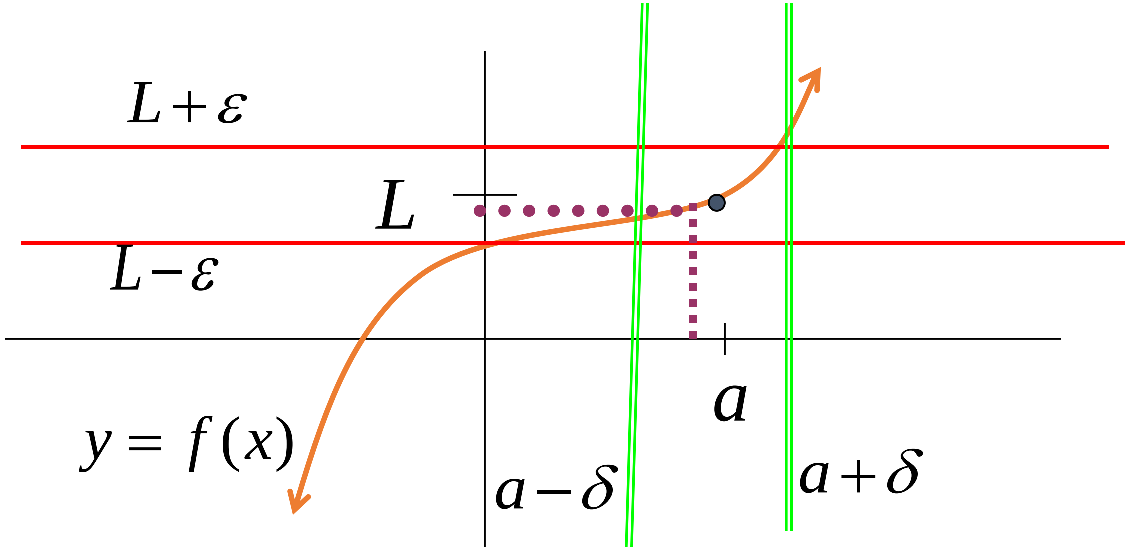
d. $\lim_{x \to 0} \left(\frac{\sqrt{1 + x} - 1}{x} \right)$ Answer : 1/2

The ε-δ Definition of Limit

We say $\lim_{x \to a} f(x) = L$ if and only if

given a positive number ε, there exists a positive δ such that

if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.



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This means that if we are given a

small interval $(L-\varepsilon, L+\varepsilon)$ centered at L,

then we can find a (small) interval $(a-\delta, a+\delta)$

such that for all $x \neq a$ in $(a-\delta, a+\delta)$,

$f(x)$ is in $(L-\varepsilon, L+\varepsilon)$.

Examples

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1. Show that $\lim_{x\to 2}(3x+4)=10.$

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

if $|x-2| < \delta$, then $|(3x+4)-10| < \varepsilon.$

But $|(3x+4)-10| = |3x-6| = 3|x-2| < \varepsilon$

if $|x-2| < \dfrac{\varepsilon}{3}$ So we choose $\delta = \dfrac{\varepsilon}{3}.$

2. Show that $\lim_{x\to 1}\dfrac{1}{x}=1.$

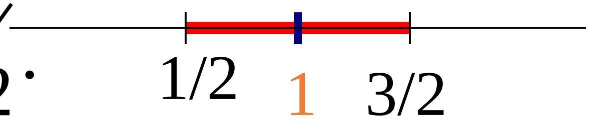
Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

if $|x-1| < \delta$, then $|{}^{1}\!/_{x} - 1| < \varepsilon.$

But $|{}^{1}\!/_{x} - 1| = \left|\dfrac{x-1}{x}\right| = \dfrac{1}{x}|x-1|.$

What do we do with the x?

If we decide $|x-1|<\frac{1}{2}$, then $\frac{1}{2}<x<\frac{3}{2}$.



And so $\frac{1}{x}<2$.

Thus $|\frac{1}{x}-1|=\frac{1}{x}|x-1|<2|x-1|$.

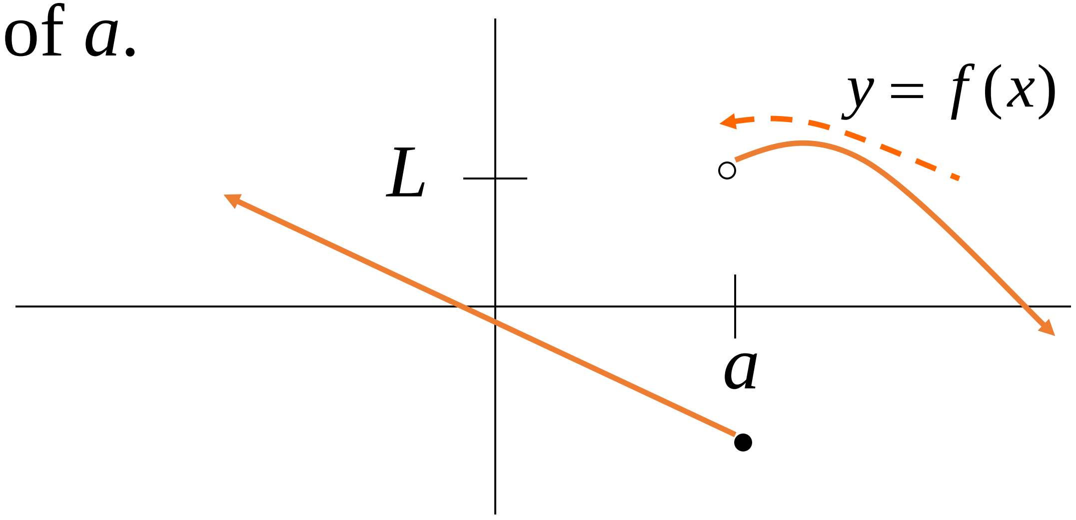
Now we choose $\delta=\min\left\{\frac{\varepsilon}{3},\frac{1}{2}\right\}$.

One-Sided Limits

The right-hand limit of $f(x)$, as x approaches a, equals L

written: $\lim_{x \to a^+} f(x) = L$

if we can make the value f (x) arbitrarily close to L by taking x to be sufficiently close to the right of a.



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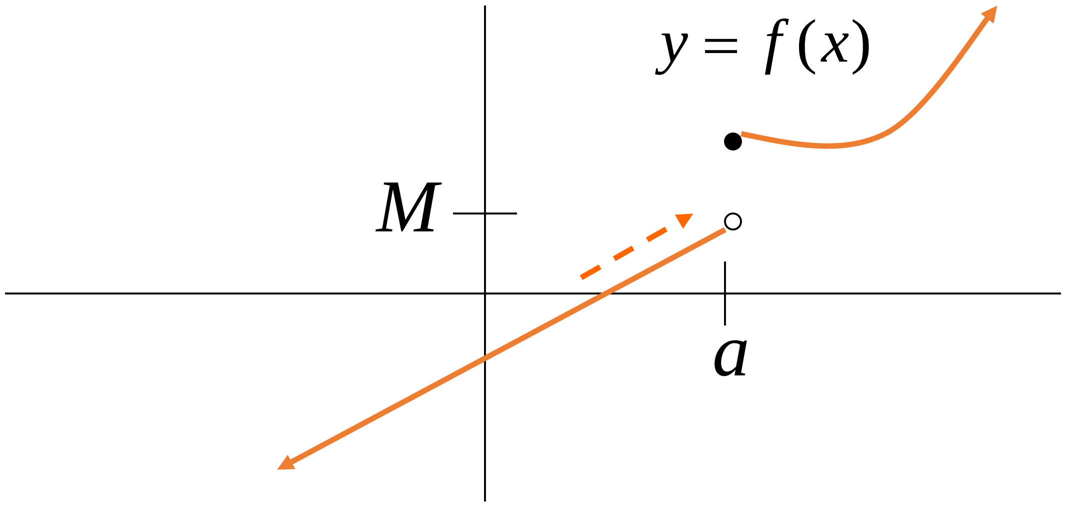
The left-hand limit of $f(x)$, as x approaches a, equals M

written: $\lim_{x \to a^-} f(x) = M$

if we can make the value f (x) arbitrarily close to L by taking x to be sufficiently close to the left of a.



Examples of One-Sided Limit

1. Given $$f(x)=\begin{cases} x^2 & \text{if } x\le 3 \\ 2x & \text{if } x>3 \end{cases}$$

Find $\lim_{x\to 3^+} f(x)$

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} 2x=6$$

Find $\lim_{x\to 3^-} f(x)$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-} x^2=9$$

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More Examples

2. Let $f(x)=\begin{cases} x+1, & \text{if } x>0 \\ x-1, & \text{if } x\le 0. \end{cases}$ Find the limits:

a) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 0+1=1$

b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x-1) = 0-1=-1$

c) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+1) = 1+1=2$

d) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x+1) = 1+1=2$

A Theorem

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L.$

This theorem is used to show a limit does not exist.

For the function $f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x \le 0. \end{cases}$

$\lim_{x \to 0} f(x)$ does not exist because $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1.$

But

$\lim_{x \to 1} f(x) = 2$ because $\lim_{x \to 1^+} f(x) = 2$ and $\lim_{x \to 1^-} f(x) = 2.$

Limit Theorems

If c is any number, $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

a) $\lim_{x\to a}\left(f(x)+g(x)\right) = L + M$

b) $\lim_{x\to a}\left(f(x)-g(x)\right) = L - M$

c) $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = L \cdot M$

d) $\lim_{x\to a}\left(f(x)/g(x) \right) = L/M$, $(M \neq 0)$

e) $\lim_{x\to a}\left(c \cdot f(x)\right) = c \cdot L$

f) $\lim_{x\to a}\left(f(x)\right)^n = L^n$

g) $\lim_{x\to a} c = c$

h) $\lim_{x\to a} x = a$

i) $\lim_{x\to a} x^n = a^n$

j) $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$, $(L > 0)$

Examples Using Limit Rule

Ex. $\lim_{x\to 3}\left(x^2+1\right)=\lim_{x\to 3}x^2+\lim_{x\to 3}1$

$=\left(\lim_{x\to 3}x\right)^2+\lim_{x\to 3}1$

$=3^2+1=10$

Ex. $\lim_{x\to 1}\frac{\sqrt{2x-1}}{3x+5}=\frac{\sqrt{\lim_{x\to 1}(2x-1)}}{\lim_{x\to 1}(3x+5)}=\frac{\sqrt{2\lim_{x\to 1}x-\lim_{x\to 1}1}}{3\lim_{x\to 1}x+\lim_{x\to 1}5}$

$=\frac{\sqrt{2-1}}{3+5}=\frac{1}{8}$

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More Examples

1. Suppose $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -2$. Find

a) $\lim_{x\to 3}\left(f(x) + g(x) \right) = \lim_{x\to 3} f(x) + \lim_{x\to 3} g(x)$

$= 4 + (-2) = 2$

b) $\lim_{x\to 3}\left(f(x) - g(x) \right) = \lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)$

$= 4 - (-2) = 6$

c) $\lim_{x\to 3}\left(\dfrac{2f(x) - g(x)}{f(x)g(x)} \right) = \dfrac{\lim_{x\to 3} 2f(x) - \lim_{x\to 3} g(x)}{\lim_{x\to 3} f(x) \cdot \lim_{x\to 3} g(x)} = \dfrac{2\cdot 4 - (-2)}{4\cdot(-2)} = \dfrac{-5}{4}$

Indeterminate Forms

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Indeterminate forms occur when substitution in the limit results in 0/0. In such cases either factor or rationalize the expressions.

Ex. $$\lim_{x \to -5} \frac{x+5}{x^2-25}$$

Notice $\frac{0}{0}$ form

$$= \lim_{x \to -5} \frac{x+5}{(x-5)(x+5)}$$

Factor and cancel common factors

$$= \lim_{x \to -5} \frac{1}{(x-5)} = \frac{1}{-10}$$

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More Examples

a) $$\lim_{x\to 9}\left(\frac{\sqrt{x}-3}{x-9}\right) = \lim_{x\to 9}\left(\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}\right)$$

$$= \lim_{x\to 9}\left(\frac{x-9}{(x-9)(\sqrt{x}+3)}\right) = \lim_{x\to 9}\left(\frac{1}{\sqrt{x}+3}\right) = \frac{1}{6}$$

b) $$\lim_{x\to -2}\left(\frac{4-x^2}{2x^2+x^3}\right) = \lim_{x\to -2}\left(\frac{(2-x)(2+x)}{x^2(2+x)}\right)$$

$$= \lim_{x\to -2}\left(\frac{2-x}{x^2}\right)$$

$$= \frac{2-(-2)}{(-2)^2} = \frac{4}{4} = 1$$

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The Squeezing Theorem

If $f(x) \le g(x) \le h(x)$ when x is near a, and if

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$

Example: Show that $\lim_{x\to 0} x^2 sin\left(\pi/x\right) = 0.$

Note that we cannot use product rule because $\lim_{x\to 0} sin\left(\pi/x\right)$ DNE!

But $-1 \le \sin\left(\pi/x\right) \le 1$ and so $-x^2 \le x^2 \sin\left(\pi/x\right) \le x^2.$

Since $\lim_{x\to 0} x^2 = \lim_{x\to 0}(-x^2) = 0,$ we use the Squeezing Theorem to conclude

$$\lim_{x\to 0} x^2 sin\left(\pi/x\right) = 0.$$

See Graph

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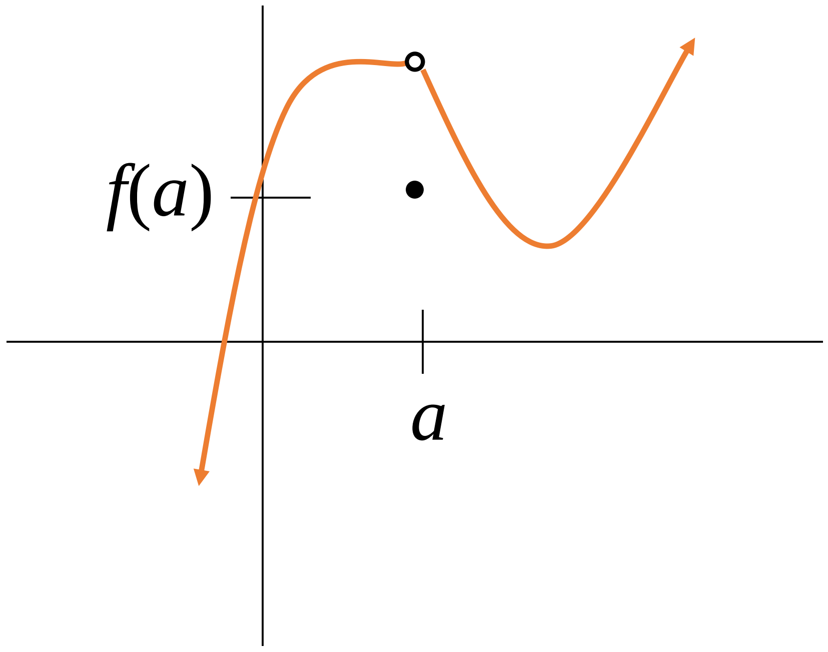
Continuity

A function f is ***continuous*** at the point $x = a$ if the following are true:

$$i)\ \ f(a)\ \text{ is defined}$$

$$ii)\ \ \lim_{x\to a} f(x)\ \text{ exists}$$



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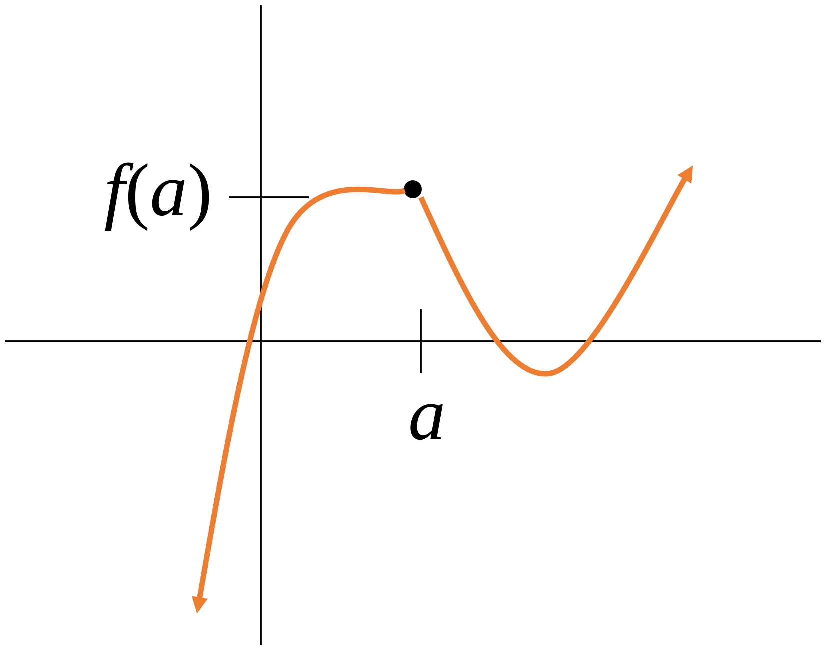
A function *f* is ***continuous*** at the point $x = a$ if the following are true:

$i)\ f(a)$ is defined

$ii)\ \lim_{x\to a} f(x)$ exists

$iii)\ \lim_{x\to a} f(x) = f(a)$



Examples

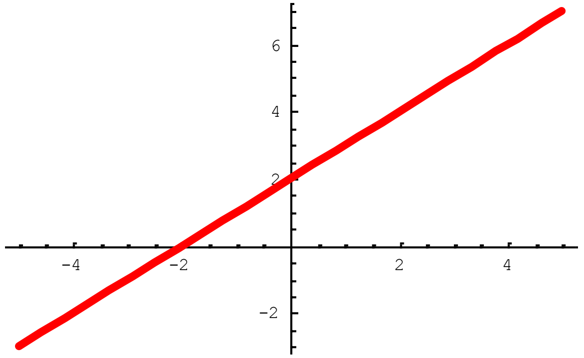
At which value(s) of x is the given function discontinuous?

1. $f(x) = x + 2$

Continuous everywhere

$$\lim_{x \to a}(x+2) = a + 2$$

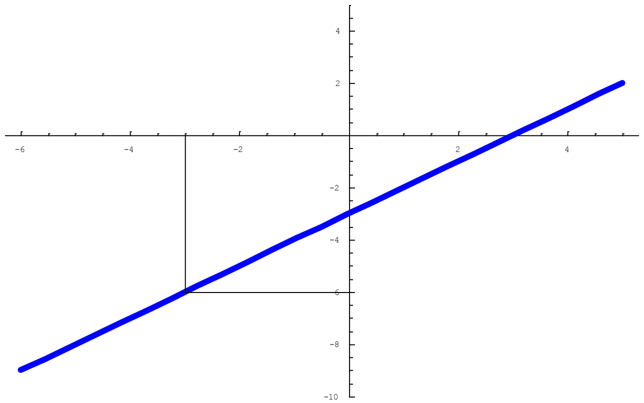
and so $\lim_{x \to a} f(x) = f(a)$



2. $g(x) = \dfrac{x^2 - 9}{x+3}$

Continuous everywhere except at $x = -3$

$g(-3)$ is undefined



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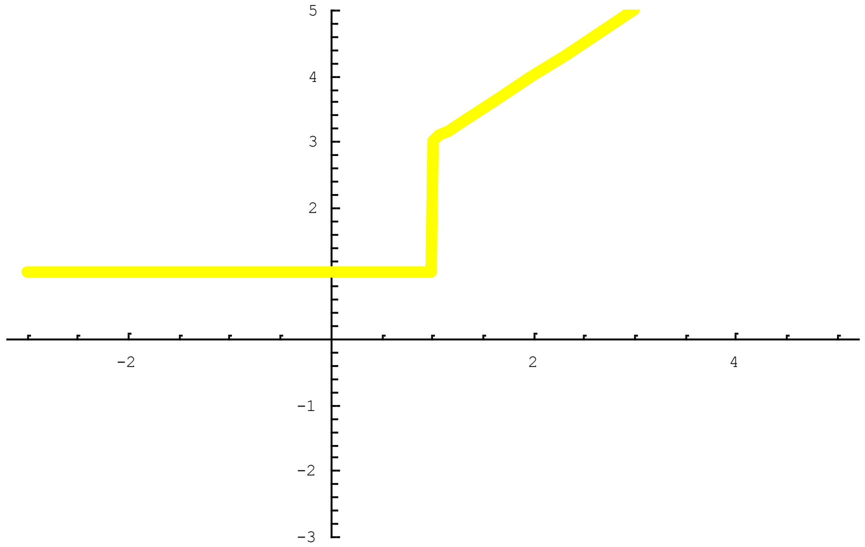
3. $h(x)=\begin{cases} x+2, \text{ if } x>1 \\ 1, \text{ if } x\le 1 \end{cases}$

$\lim\limits_{x\to 1^-} h(x)=1$ and $\lim\limits_{x\to 1^+} h(x)=3$

Thus *h* is not cont. at *x*=*1*.

h is continuous everywhere else

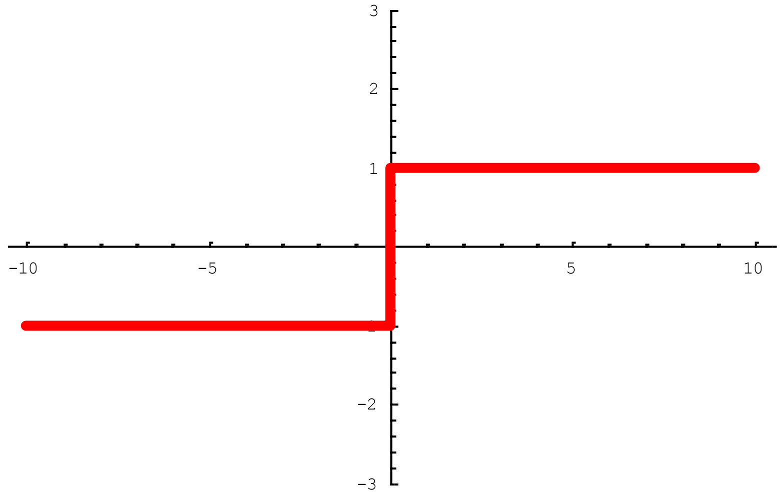


4. $F(x)=\begin{cases} -1, \text{ if } x\le 0 \\ 1, \text{ if } x>0 \end{cases}$

$\lim\limits_{x\to 0^+} F(x)=1$ and $\lim\limits_{x\to 0^-} F(x)=-1$

Thus *F* is not cont. at $x=0$.

F is continuous everywhere else



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Continuous Functions

If f and g are continuous at $x = a$, then

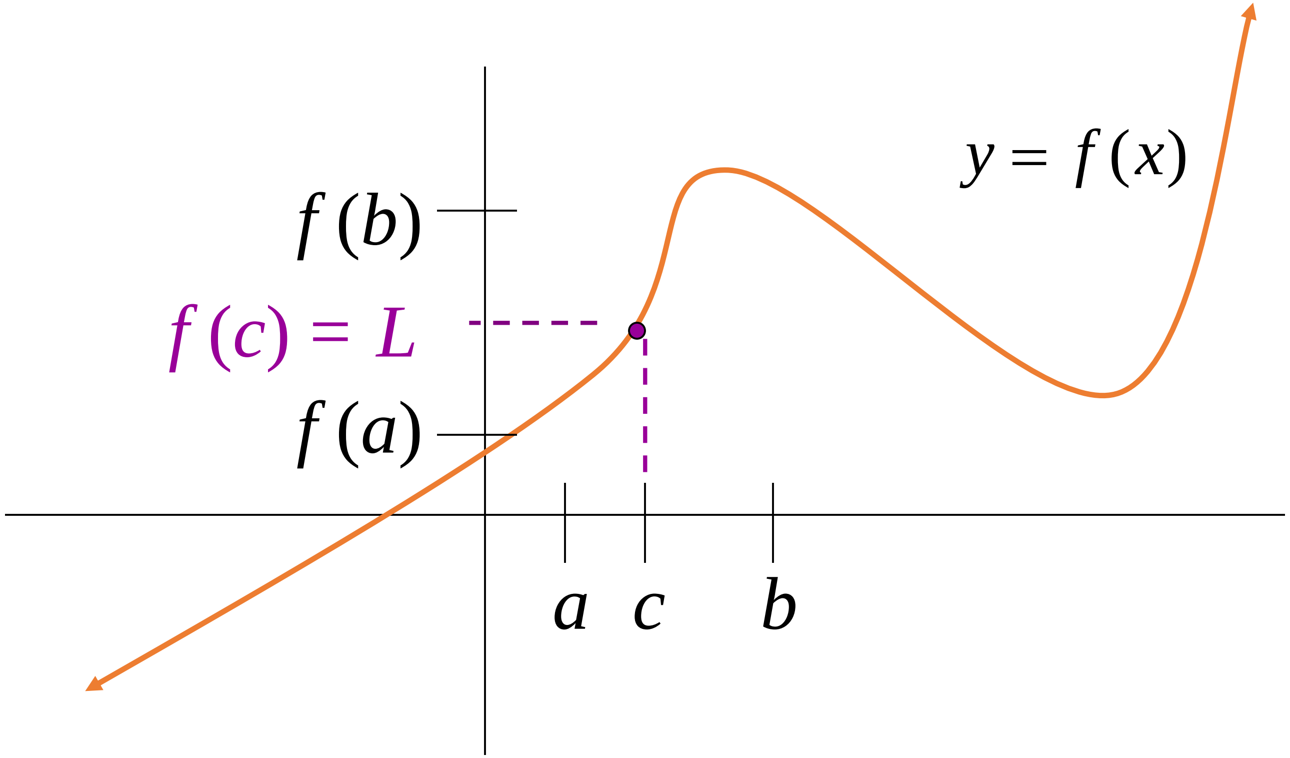
$f \pm g,\ \ fg,\ $ and f/g $\left(g(a) \neq 0\right)$ are continuous at $x = a$

A polynomial function $y = P(x)$ is continuous at every point x.

A rational function $R(x) = p(x)/q(x)$ is continuous at every point x in its domain.

Intermediate Value Theorem

If f is a continuous function on a closed interval $[a, b]$ and L is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = L$.



Example

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Given $f(x) = 3x^2 - 2x - 5$,

Show that $f(x) = 0$ has a solution on $[1, 2]$.

$$f(1) = -4 < 0$$

$$f(2) = 3 > 0$$

$f(x)$ is continuous (polynomial) and since $f(1) < 0$ and $f(2) > 0$, by the Intermediate Value Theorem there exists a c on [1, 2] such that $f(c) = 0$.

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Limits at Infinity

For all $n > 0$, $\displaystyle \lim_{x\to\infty}\frac{1}{x^n} = \lim_{x\to-\infty}\frac{1}{x^n} = 0$

provided that $\dfrac{1}{x^n}$ is defined.

Ex. $$\lim_{x\to\infty}\frac{3x^2+5x+1}{2-4x^2} = \lim_{x\to\infty}\frac{3+\frac{5}{x}+\frac{1}{x^2}}{\frac{2}{x^2}-4}$$

Divide by x^2

$$= \frac{\lim\limits_{x\to\infty}3+\lim\limits_{x\to\infty}\left(\frac{5}{x}\right)+\lim\limits_{x\to\infty}\left(\frac{1}{x^2}\right)}{\lim\limits_{x\to\infty}\left(\frac{2}{x^2}\right)-\lim\limits_{x\to\infty}4} = \frac{3+0+0}{0-4} = -\frac{3}{4}$$

More Examples

1. $$\lim_{x\to\infty}\left(\frac{2x^3-3x^2+2}{x^3-x^2-100x+1}\right)=\lim_{x\to\infty}\left(\frac{\frac{2x^3}{x^3}-\frac{3x^2}{x^3}+\frac{2}{x^3}}{\frac{x^3}{x^3}-\frac{x^2}{x^3}-\frac{100x}{x^3}+\frac{1}{x^3}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{2-\frac{3}{x}+\frac{2}{x^3}}{1-\frac{1}{x}-\frac{100}{x^2}+\frac{1}{x^3}}\right)$$

$$=\frac{2}{1}=2$$

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2. $$\lim_{x\to\infty}\left(\frac{4x^2-5x+21}{7x^3+5x^2-10x+1}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4x^2}{x^3}-\frac{5x}{x^3}+\frac{21}{x^3}}{\frac{7x^3}{x^3}+\frac{5x^2}{x^3}-\frac{10x}{x^3}+\frac{1}{x^3}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4}{x}-\frac{5}{x^2}+\frac{21}{x^3}}{7+\frac{5}{x}-\frac{10}{x^2}+\frac{1}{x^3}}\right)$$

$$=\frac{0}{7}$$

$$=0$$

3. $$\lim_{x\to\infty}\left(\frac{x^2+2x-4}{12x+31}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{x^2}{x}+\frac{2x}{x}-\frac{4}{x}}{\frac{12x}{x}+\frac{31}{x}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x+2-\frac{4}{x}}{12+\frac{31}{x}}\right)$$

$$=\frac{\infty+2}{12}$$

$$=\infty$$

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4. $$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

$$=\lim_{x\to\infty}\left(\frac{\left(\sqrt{x^2+1}-x\right)}{1}\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right)$$

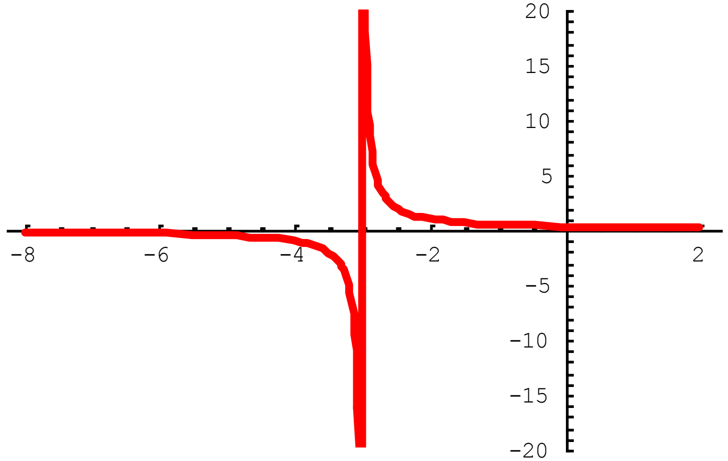
$$=\lim_{x\to\infty}\left(\frac{1}{\sqrt{x^2+1}+x}\right)$$

$$=\frac{1}{\infty+\infty}=\frac{1}{\infty}=0$$

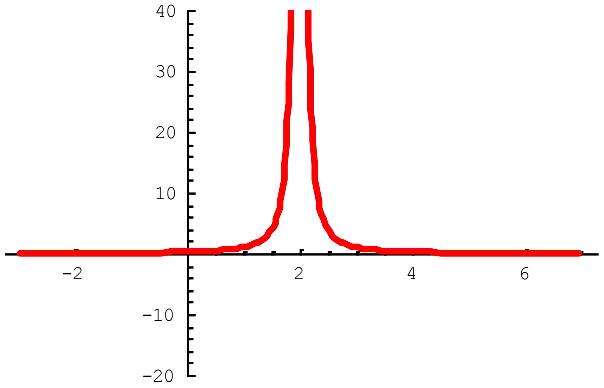
Infinite Limits

For all $n > 0$,

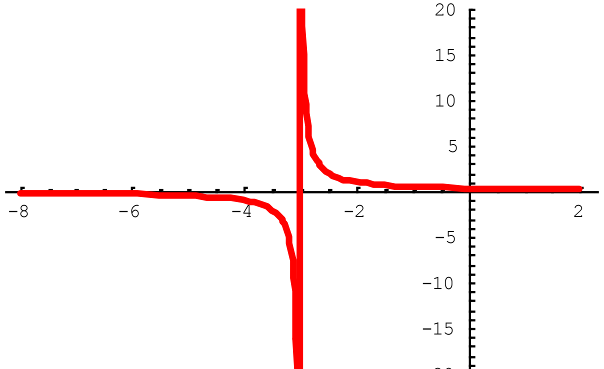
$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = \infty$$



$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = \infty \text{ if } n \text{ is even}$$



$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty \text{ if } n \text{ is odd}$$



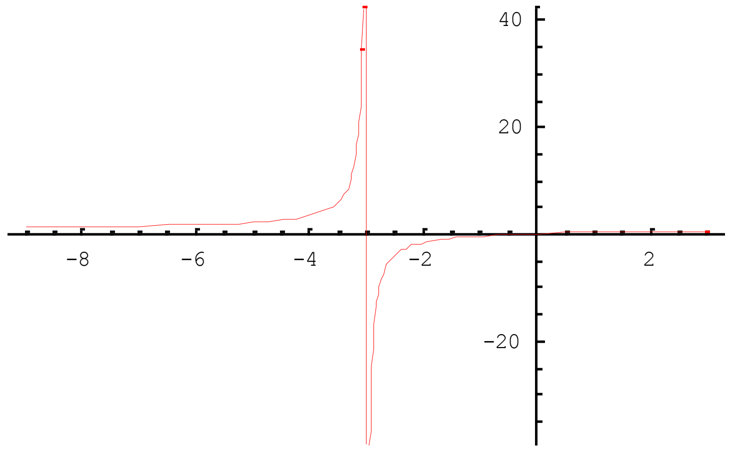
[More Graphs]

Examples

Find the limits

1. $$\lim_{x\to 0^+}\left(\frac{3x^2+2x+1}{2x^2}\right) = \lim_{x\to 0^+}\left(\frac{3+{}^2\!/_x+{}^1\!/_{x^2}}{2}\right) = \frac{3+\infty+\infty}{2} = \infty$$

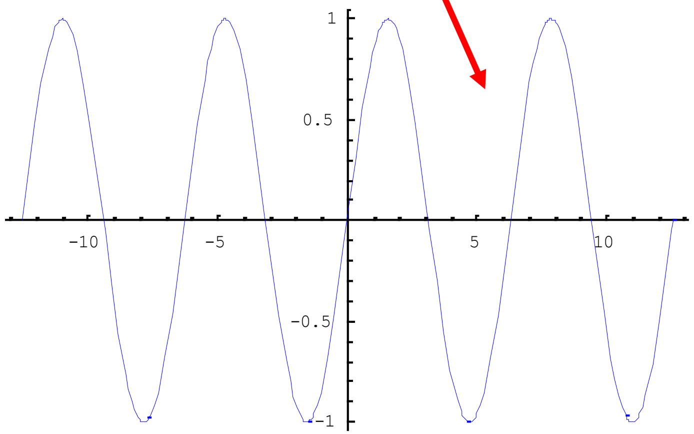
2. $$\lim_{x\to -3^+}\left(\frac{2x+1}{2x+6}\right) = \lim_{x\to -3^+}\left(\frac{2x+1}{2(x+3)}\right) = -\infty$$

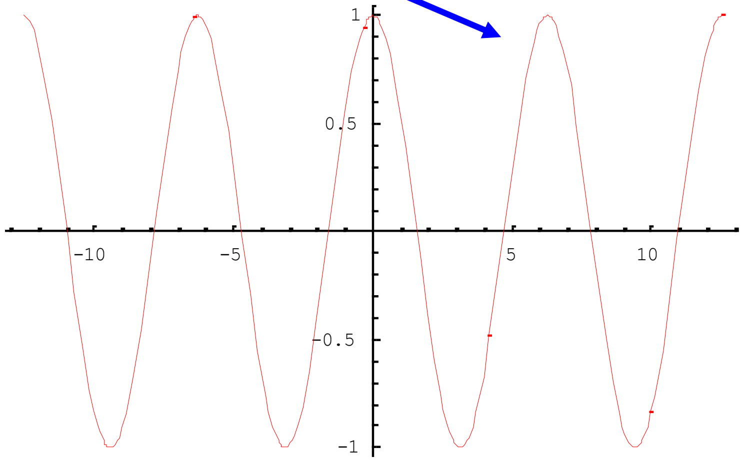


Limit and Trig Functions

From the graph of trigs functions

$$f(x) = \sin x \text{ and } g(x) = \cos x$$





we conclude that they are continuous everywhere

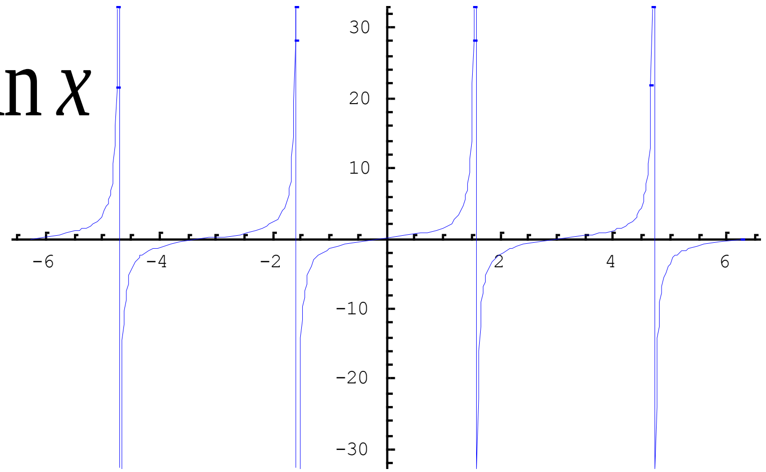
$$\lim_{x\to c} \sin x = \sin c \text{ and } \lim_{x\to c} \cos x = \cos c$$

Tangent and Secant

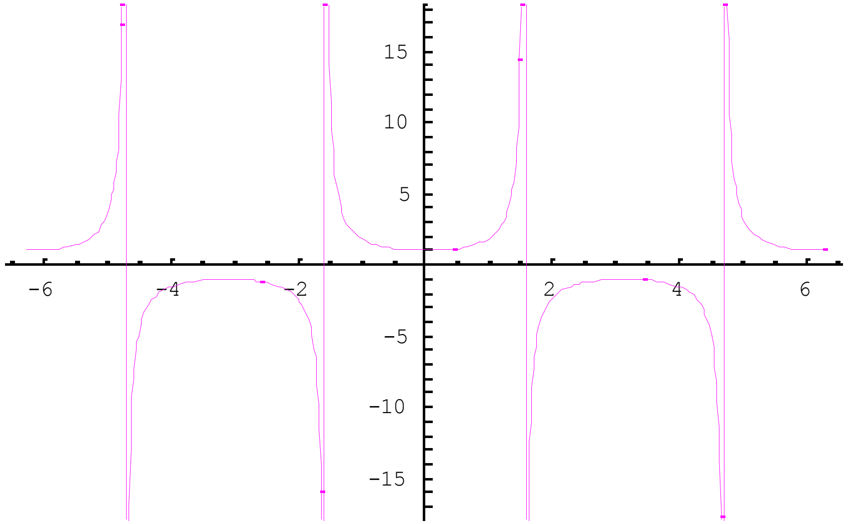
Tangent and secant are continuous everywhere in their domain, which is the set of all real numbers

$$x \neq \pm \pi/2,\ \pm 3\pi/2,\ \pm 5\pi/2,\ \pm 7\pi/2, \ldots$$

$y = \tan x$



$y = \sec x$



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Examples

a) $\lim_{x \to (\pi/2)^+} \sec x = -\infty$

b) $\lim_{x \to (\pi/2)^-} \sec x = \infty$

c) $\lim_{x \to (-3\pi/2)^+} \tan x = -\infty$

d) $\lim_{x \to (-3\pi/2)^-} \tan x = \infty$

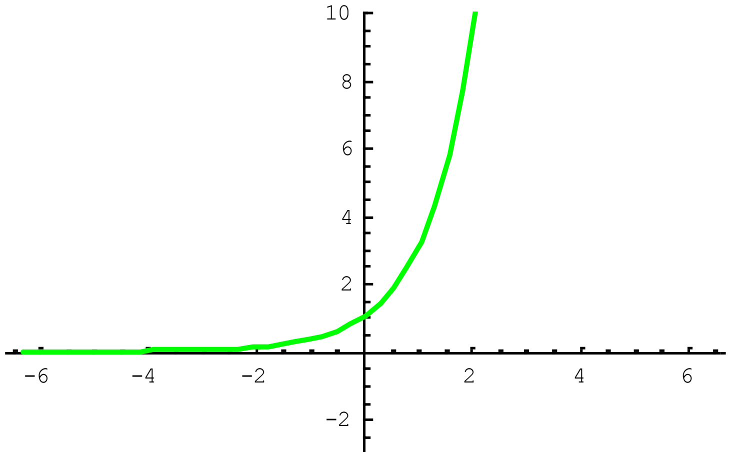
e) $\lim_{x \to \pi^-} \cot x = -\infty$

f) $\lim_{x \to \pi/4} \tan x = 1$

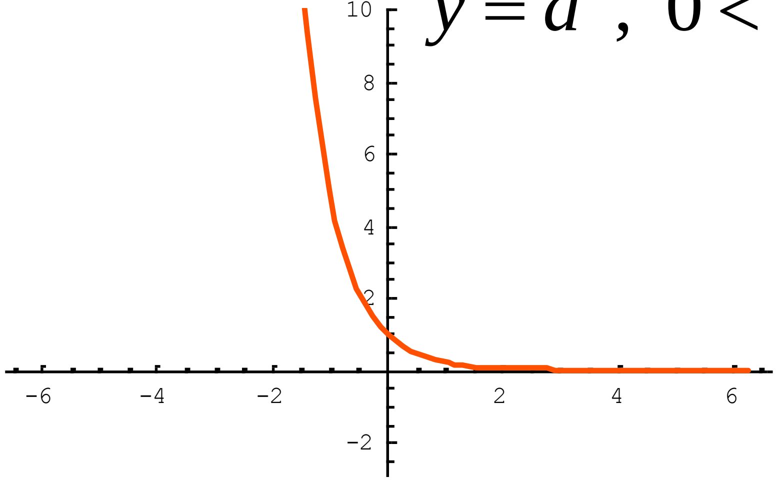
g) $\lim_{x \to (-3\pi/2)} \cot x = \lim_{x \to (-3\pi/2)} \dfrac{\cos x}{\sin x} = \dfrac{0}{1} = 0$

Limit and Exponential Functions

$y = a^x, \ a > 1$



$y = a^x, \ 0 < a < 1$



The above graph confirm that exponential functions are continuous everywhere.

$$\lim_{x \to c} a^x = a^c$$

Asymptotes

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if eihter

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L.$$

The line $x = c$ is called a vertical asymptote of the curve $y = f(x)$ if eihter

$$\lim_{x\to c^-} f(x) = \pm\infty \text{ or } \lim_{x\to c^+} f(x) = \pm\infty.$$

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Examples

Find the asymptotes of the graphs of the functions

1. $f(x) = \dfrac{x^2+1}{x^2-1}$

(i) $\lim\limits_{x \to 1^-} f(x) = -\infty$

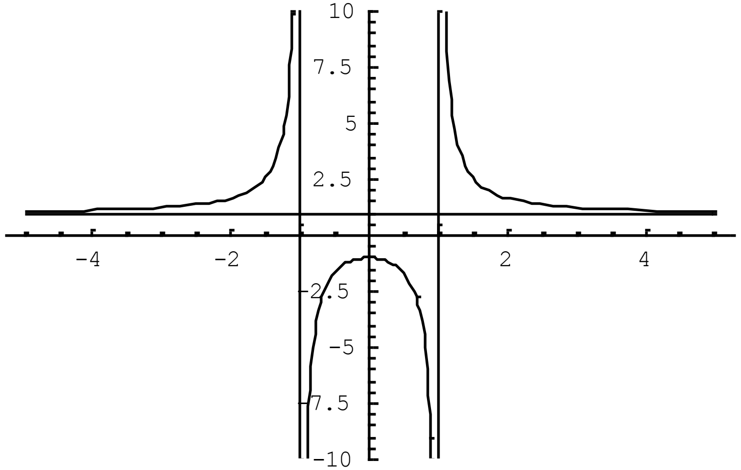
Therefore the line $x = 1$ is a vertical asymptote.

(ii) $\lim\limits_{x \to -1^-} f(x) = +\infty.$

Therefore the line $x = -1$ is a vertical asymptote.

(iii) $\lim\limits_{x \to \infty} f(x) = 1.$

Therefore the line $y = 1$ is a horizonatl asymptote.



2. $f(x) = \dfrac{x-1}{x^2-1}$

(i) $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}\left(\dfrac{x-1}{x^2-1}\right)$

$= \lim\limits_{x\to 1}\left(\dfrac{x-1}{(x-1)(x+1)}\right) = \lim\limits_{x\to 1}\left(\dfrac{1}{x+1}\right) = \dfrac{1}{2}.$

Therefore the line $x=1$

is NOT a vertical asymptote.

(ii) $\lim\limits_{x\to -1^+} f(x) = +\infty.$

Therefore the line $x=-1$

is a vertical asymptote.

(iii) $\lim\limits_{x\to\infty} f(x) = 0.$

Therefore the line $y=0$

is a horizonatl asymptote.